

Aspects of non-abelian gauge dynamics in two-dimensional $\mathcal{N} = (2, 2)$ theories

Kentaro Hori

*University of Toronto,
Toronto, Ontario, Canada
E-mail: hori@physics.utoronto.ca*

David Tong

*DAMTP, University of Cambridge,
Wilberforce Road, Cambridge, U.K.
E-mail: d.tong@damtp.cam.ac.uk*

ABSTRACT: We study various aspects of $\mathcal{N} = (2, 2)$ supersymmetric non-Abelian gauge theories in two dimensions, with applications to string vacua. We compute the Witten index of $SU(k)$ SQCD with $N > 0$ flavors with twisted masses; the result is presented as the solution to a simple combinatoric problem. We further claim that the infra-red fixed point of $SU(k)$ gauge theory with N massless flavors is non-singular if (k, N) passes a related combinatoric criterion. These results are applied to the study of a class of $U(k)$ linear sigma models which, in one phase, reduce to sigma models on Calabi-Yau manifolds in Grassmannians. We show that there are multiple singularities in the middle of the one-dimensional Kähler moduli space, in contrast to the Abelian models. This result precisely matches the complex structure singularities of the proposed mirrors. In one specific example, we study the physics in the other phase of the Kähler moduli space and find that it reduces to a sigma model for a second Calabi-Yau manifold which is not birationally equivalent to the first. This proves a mathematical conjecture of Rødland.

KEYWORDS: Supersymmetric gauge theory, Conformal Field Models in String Theory, Field Theories in Lower Dimensions.

Contents

1. Introduction	1
2. Splitting the conifold singularity	4
2.1 The models	5
2.2 Abelian theories	7
2.3 Non-abelian theories	8
2.4 Comparison with mirror symmetry conjecture	11
3. The Witten index for two dimensional SQCD	12
3.1 A first attempt at counting	13
3.2 The absence of contributions from strong coupling regimes	15
3.3 Varying the twisted masses	17
3.4 Complex masses for SU(2) theories	19
4. Infra-red dynamics of SU(k) gauge theories	21
4.1 IR central charge	21
4.2 The singularity criterion	23
4.3 $N = k + 1$: duality with free SCFT and LG models	24
4.4 Interlude — The U(k) linear sigma model revisited	27
4.5 RG flow and duality	29
4.6 A test of the IR singularity	31
5. The glop transition	33
5.1 Rødland's conjecture	33
5.2 The genericity condition	34
5.3 The physics at $r \ll 0$	34
5.4 A comment on D-brane categories	36
5.5 Generalization	36

1. Introduction

The purpose of this paper is to study the quantum dynamics of two dimensional $\mathcal{N} = (2, 2)$ supersymmetric gauge theories with non-Abelian gauge groups. We will discover a number of interesting features that are novel to non-Abelian theories and do not occur for Abelian gauge groups.

Our motivation for this work arose from a number of mathematical conjectures concerning the moduli spaces of Calabi-Yau manifolds embedded in Grassmannians [1, 2]. The

properties of these moduli spaces exhibit qualitative differences from those for Calabi-Yau manifolds in toric varieties. The physicist's tool to study these conjectures is the linear sigma model, a gauge theory designed to flow in the infra-red to the desired Calabi-Yau target space [3]. While Calabi-Yau manifolds in toric varieties may be constructed using only Abelian gauge theories, to build Calabi-Yau in Grassmannians one must necessarily work with non-Abelian gauge groups.

A typical theory of interest has a $U(k)$ gauge group with N chiral multiplets Φ in the fundamental representation and a number of chiral multiplets P which are (negatively) charged under the central $U(1) \subset U(k)$. The P and Φ fields are coupled through a superpotential. The low-energy physics of this model depends on the value of the Fayet-Iliopolous (FI) parameter r associated with the central $U(1) \subset U(k)$. For $r \gg 0$, the fundamental fields Φ gain an expectation value, completely breaking the $U(k)$ gauge group and ensuring that one does not have to contend with strongly coupled non-Abelian dynamics. It is in this regime that the gauge theory reduces to a sigma-model on the compact Calabi-Yau 3-fold of interest. However, to fully understand the moduli space of the Calabi-Yau one must also study the theory at small values of r and at $r \ll 0$. In these regions, the full force of the non-Abelian gauge dynamics is at play. For example, at $r = 0$ there is a locus on which the $U(k)$ gauge group is unbroken and a k dimensional non-compact Coulomb branch with an unbroken $U(1)^k$ emerges. Similarly, for $r \ll 0$, the P fields gain an expectation value, breaking $U(k)$ to $SU(k)$.

We are therefore invited to study the dynamics of $U(k)$ and $SU(k)$ gauge theories coupled to fundamental chiral multiplets. We answer basic questions concerning the vacuum structure of these theories: How many supersymmetric ground states are there? Under what circumstances are the ground states normalizable? When does the theory flow to a non-trivial fixed point? In answering these questions, we find several new phenomena which do not occur for Abelian models. We now give a summary of the main results:

The Witten index: although the Witten index [4] for $U(k)$ theories in two dimensions is easy to compute, to our knowledge the calculation for $SU(k)$ theories has not appeared in the literature. In section 3 we derive the Witten index for $\mathcal{N} = (2, 2)$ $SU(k)$ supersymmetric QCD with N flavors, each endowed with a twisted mass [5]. As we will explain, the non-compact Coulomb branch of this theory is lifted by quantum effects, leaving behind isolated, supersymmetric vacua. The Witten index is given by the solution to a simple combinatoric problem: Find k distinct N -th roots of unity, modulo overall scaling, whose sum is non-zero. In particular, there is no supersymmetric ground state for $1 \leq N \leq k$ and there is exactly one for $N = k + 1$. We list below the index for low values of k and N .

IR Dynamics of $SU(k)$ gauge theories: in section 4, we study the infra-red dynamics of $SU(k)$ gauge theories with N *massless* fundamental chiral multiplets. Such theories are expected to flow to superconformal field theories (SCFTs) in the infra-red limit. For example, if the superpotential is a homogeneous polynomial of degree d in the baryon operators, the gauge theory flows to a SCFT with central charge $\hat{c} = c/3 = N(k - 2/d) - k^2 + 1$. However, the potential existence of a Coulomb branch — a non-compact, flat direction in field space — means that the ground state wavefunction may spread, rendering

$k \setminus N$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	0	0	1	1	2	2	3	3	4	4	5	5	6	6	7
3	0	0	0	1	2	3	5	7	9	12	15	18	22	26	30
4	0	0	0	0	1	2	5	8	14	20	30	40	55	70	91
5	0	0	0	0	0	1	3	7	14	25	42	65	99	143	200
6	0	0	0	0	0	0	1	3	9	20	42	75	132	212	333
7	0	0	0	0	0	0	0	1	4	12	30	65	132	245	429

Table 1: The Witten index for $SU(k)$ SQCD with N massive flavors

the conformal field theory singular. Such behavior is seen at the conifold point of $\mathcal{N} = (2, 2)$ Abelian theories [3] and at the special point of $\mathcal{N} = (4, 4)$ SQED [6–8]. We propose that the $SU(k)$ theory suffers from such a singularity if and only if there exist k distinct N -th roots of unity that sum to zero. For example, the low energy theory of an $SU(k)$ theory with $N = k + 1$ fundamentals is always non-singular and is described by the N baryon operators as the independent variables. In particular, $SU(k)$ SQCD with $N = k + 1$ massless flavors flows to a free conformal field theory with $\hat{c} = N$. We also propose an infra-red duality between the $SU(k)$ gauge theory and the $SU(N - k)$ gauge theory, both with N fundamentals and degree d superpotential for the baryons. The duality is proved for the case $\frac{2N}{Nk - k^2 + 1} < d \leq N$.

Splitting the conifold singularity: for the class of $U(k)$ linear sigma models described above, we find that there are typically multiple singular points in the middle of the one-dimensional Kähler moduli space parameterized by the FI-theta parameter $t = r - i\theta$. This behavior is in contrast to $U(1)$ theories which always have exactly one singular point in a one-dimensional Kähler moduli space. These multiple singularities, which arise from a quantum mechanical splitting of the classical singularity, are the topic of section 2. The analysis is based on the the quantum potential on the Coulomb branch [3, 9]. If there are N fundamental chiral multiplets, and several fields charged under the $U(1) \subset U(k)$, the number of singular points coincides with the Witten index for $SU(k)$ SQCD with N massive flavors. At each of these points, a one-dimensional subspace in the k -dimensional Coulomb branch becomes a truly flat direction. For the models corresponding to Calabi-Yau three-folds, the result precisely matches the singularities of the complex structure moduli space of the proposed mirrors, giving a strong support to the mirror symmetry conjecture of [1].

The glop transition: one of the original motivations for our work is the conjecture by Rødland [2] that two inequivalent compact Calabi-Yau three-folds X and Y sit on the same one-dimensional complexified Kähler moduli space. X is a submanifold in the Grassmannian $G(2, 7)$ while Y is an *incomplete* intersection in \mathbb{CP}^6 , referred to as the Pfaffian Calabi-Yau. We wish to understand this claim from a purely quantum field theoretic point of view. The natural linear sigma-model for the Calabi-Yau X is of the type described above: it is a $U(2)$ gauge theory with 7 fundamental chiral multiplets Φ_i and a further 7 chiral multiplets P_i , transforming in the \det^{-1} representation (i.e. charged only under the

central $U(1) \subset U(2)$). These chiral multiplets are coupled through the superpotential

$$W = \sum_{i,j,k=1}^7 A_i^{jk} P^i (\Phi_j^1 \Phi_k^2 - \Phi_j^2 \Phi_k^1) \quad (1.1)$$

where A_i^{jk} are generic coefficients that are anti-symmetric in the upper indices. In the regime $r \gg 0$, the manifold X appears, sitting at the bottom of the scalar potential of this theory. Moving towards the center of the Kähler moduli space, around $r \sim 0$, one finds three singular points — this can be read from the $(k, N) = (2, 7)$ entry of table 1 for the Witten index.

The regime $r \ll 0$ is the focus of section 5, where we make use of many results from previous sections. The D-term equations force the P 's to span \mathbb{CP}^6 and the low energy theory includes an unbroken $SU(2)$ gauge theory with $N = 7$ flavors Φ_i . The superpotential (1.1) provides a complex mass matrix $A(P)^{jk} = \sum_{i=1}^7 A_i^{jk} P^i$ for these flavors which varies as we move around \mathbb{CP}^6 . The antisymmetric matrix $A(P)$ has rank 6 at a generic point P of \mathbb{CP}^6 but it degenerates to rank 4 at a locus of codimension three. This rank 4 locus is precisely the location of the Pfaffian Calabi-Yau Y . The number of massless flavors is $N = 1$ on the rank 6 domain but it jumps to $N = 3$ on the rank 4 locus. Looking at the Witten index in table 1, we see that supersymmetry is broken on the rank 6 domain and the low energy theory localizes on the rank 4 locus. In addition to the tangent modes, the low energy theory on the rank 4 locus contains an $SU(2)$ gauge multiplet, $N = 3$ massless fundamentals, and three singlets from the transverse modes of P . However, since any $SU(k)$ gauge theory with $N = k + 1$ fundamentals is described at low energies by the N independent baryon variables, this extra sector with interaction (1.1) flows to a Landau-Ginzburg model with a non-degenerate and quadratic superpotential for the three baryons and the three transverse modes. It has a unique supersymmetric ground state with a mass gap, and hence only the tangent modes to the rank 4 locus remain in the deep infra-red limit. In this way we obtain the supersymmetric non-linear sigma model on the Pfaffian Calabi-Yau Y . This provides a field theoretic proof of Rødland's conjecture [2]. Note that the Calabi-Yau Y does not arise from simply restricting to the zeroes of a classical potential, but requires us to make use of the strongly coupled non-Abelian gauge dynamics in a novel fashion.

The transition from the X phase to the Y phase is smooth within the conformal field theory. This is similar to the transition from large volume Calabi-Yau to a Landau-Ginzburg orbifold or, perhaps more closely, to the familiar flop transition [3, 10]. However, our example exhibits one crucial property that distinguishes it from the flop: X and Y are *not* birationally equivalent. For this reason we refer to this new type of topology changing transition as a Grassmannian flop, or *glop*.

2. Splitting the conifold singularity

In this section we discuss the phenomenon of the quantum splitting of conifold singularities for Calabi-Yau manifolds embedded in Grassmannians. The basic physics can already be

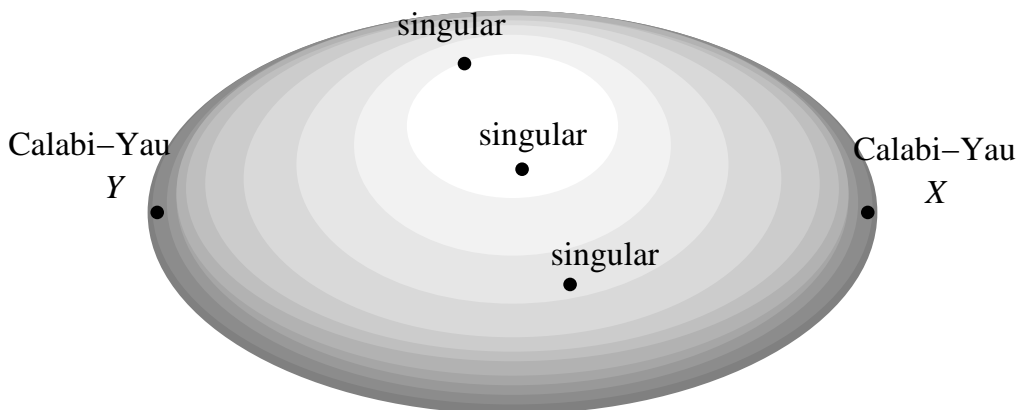


Figure 1: The Kähler Moduli Space of the IR fixed points of the theory with superpotential (1.1). It has two large volume limits and three singularities

seen in non-compact Calabi-Yau, constructed from line-bundles over Grassmannians. We start with a description of the relevant gauged linear sigma models, before presenting a derivation of our result. We then compare these results with predictions from mirror symmetry for compact Calabi-Yau 3-folds and find agreement in all cases.

2.1 The models

Throughout this paper, our main tool will be the gauged linear sigma model, a gauge theory designed to flow in the infra-red to a non-linear sigma model on a target space \mathcal{M} which arises as the classical vacuum moduli space of the theory [3].

We will work with a $U(k)$ gauge group whose field strength F_{01} lives in a twisted chiral multiplet Σ . Our notation is canonical and follows [3, 11],

$$\Sigma = \sigma + \theta^+ \bar{\lambda}_+ + \bar{\theta}^- \lambda_+ + \theta^+ \bar{\theta}^- (D - iF_{01}) + \dots \quad (2.1)$$

The gauge field couples to N chiral multiplets Φ_i transforming in the fundamental representation \mathbf{k} . Each has a component expansion,

$$\Phi_i = \phi_i + \theta^+ \psi_{i+} + \theta^- \psi_{i-} + \theta^+ \theta^- F_i + \dots \quad i = 1, \dots, N \quad (2.2)$$

To ensure that the theory flows to an interacting conformal field theory — or, equivalently, to ensure that \mathcal{M} is Calabi-Yau — we need to add further matter to cancel the axial $U(1)_A$ anomaly [3]. This can be achieved in at least two different ways: we could simply introduce N companion chiral fields $\tilde{\Phi}_i$, each transforming in the anti-fundamental representation $\bar{\mathbf{k}}$ of the gauge group. However, a more interesting possibility is to instead add S chiral multiplets P^α , each transforming in the \det^{-q_α} representation for some choice of integers q_α . This means that the field P^α is charged only under the central $U(1) \subset U(k)$ so that for $g \in U(k)$, $P^\alpha \rightarrow (\det^{-q_\alpha} g) P^\alpha$ or, infinitesimally, for $g = 1 + \epsilon$, $\delta P^\alpha = -q_\alpha (\text{Tr} \epsilon) P^\alpha$. We write

$$P^\alpha = p^\alpha + \theta^+ \chi_+^\alpha + \theta^- \chi_-^\alpha + \theta^+ \theta^- F^\alpha + \dots \quad (2.3)$$

The condition for $U(1)_A$ axial anomaly cancellation is

$$N \text{Tr}_{\mathbf{k}} F_A + \sum_{\alpha=1}^S \text{Tr}_{\det^{-q_\alpha}} F_A = \left(N - \sum_{\alpha=1}^S q_\alpha \right) \text{Tr}_{\mathbf{k}} F_A = 0, \quad (2.4)$$

which requires $\sum_{\alpha=1}^S q_\alpha = N$.

The $U(k)$ adjoint valued D -term for the theory reads,

$$D_b^a = e^2 \left(\sum_{i=1}^N \phi_i^a \phi_{bi}^\dagger - \sum_{\alpha=1}^S q_\alpha |p^\alpha|^2 \delta_b^a - r \delta_b^a \right) \quad a, b = 1, \dots, k. \quad (2.5)$$

where r is the Fayet-Iliopoulos (FI) parameter and e^2 the gauge coupling constant. In the absence of a superpotential, the vacuum moduli space is $\mathcal{M} \cong \{D = 0\}/U(k)$, a non-compact Calabi-Yau manifold which, for $r > 0$, is the sum of S line bundles over the Grassmannian $G(k, N)$ of k -planes in \mathbb{C}^N : $\mathcal{M} \cong \bigoplus_{\alpha} \mathcal{O}(q_\alpha) \rightarrow G(k, N)$.

We can construct compact Calabi-Yau manifolds $X \subset \mathcal{M}$ through the introduction of a superpotential for the chiral multiplets [3]. We consider gauge invariant superpotentials of the form,

$$W = \sum_{\alpha=1}^S P^\alpha G_\alpha(B), \quad (2.6)$$

where G_α is a polynomial of degree q_α in the Plücker coordinates (i.e. baryonic variables)

$$B_{i_1 \dots i_k} = \epsilon_{a_1 \dots a_k} \Phi_{i_1}^{a_1} \dots \Phi_{i_k}^{a_k} \quad (2.7)$$

These provide homogeneous coordinates on the projective space $\mathbb{P}(\wedge^k \mathbb{C}^N) \cong \mathbb{C}\mathbb{P}^{\binom{N}{k}-1}$ in which the Grassmannian is embedded. For $r \gg 0$, provided certain genericity conditions on G_α hold, the low-energy theory is the non-linear sigma model on the compact Calabi-Yau $X_{q_1, \dots, q_S} \subset G(k, N)$ of dimension $Nk - k^2 - S$, defined by the intersection of hypersurfaces $G_\alpha = 0$ with the Grassmannian $G(k, N)$ living at $p^\alpha = 0$. We will discuss the precise genericity conditions on G_α for a specific example in section 5.

If we restrict attention to 3-folds, the above construction yields only a handful of compact Calabi-Yau manifolds. Indeed, the dimensionality condition $Nk - k^2 - S = 3$ can be written as

$$(k-1)(N-k-1) = 4 + S - N,$$

but the right hand side is at most 4 since we have $S \leq N$ from the Calabi-Yau condition $N = \sum_{\alpha=1}^S q_\alpha$. We list the solutions below, together with their relevant Hodge numbers (taken from [1]). The sole Kähler modulus of each manifold X is inherited from the Grassmannian in which X is embedded. The moduli space of the complexified Kähler class is parameterized by the complex combination of the FI parameter r and the theta angle θ .

$$t = r - i\theta. \quad (2.8)$$

For generic t , the conformal field theory on X has well-defined correlation functions. However at certain values of t , the correlation functions diverge. From the perspective of the

X	$h^{1,1}(X)$	$h^{2,1}(X)$
$X_4 \subset G(2, 4)$	1	89
$X_{1,1,3} \subset G(2, 5)$	1	76
$X_{1,2,2} \subset G(2, 5)$	1	61
$X_{1,1,1,1,2} \subset G(2, 6)$	1	59
$X_{1,\dots,1} \subset G(2, 7)$	1	50
$X_{1,\dots,1} \subset G(3, 6)$	1	49

Table 2: 3-fold complete intersections in Grassmannians.

gauge theory, this divergence can be traced to the emergence of a new massless direction in field space — the Coulomb branch — into which the ground state wavefunctions spread [3]. Let us firstly recall how one sees the emergence of the Coulomb branch in Abelian gauge theories.

2.2 Abelian theories

Consider a $U(1)$ gauge theory with N chiral multiplets of charge Q_i , $i = 1, \dots, N$. The criterion for conformal invariance is $\sum_i Q_i = 0$. Since the location of the singularities in the Kähler moduli space does not depend on the complex structure moduli, we may perform the analysis for the situation with vanishing superpotential and the corresponding non-compact Calabi-Yau. In this case the classical vacuum energy is given by,

$$V = \frac{e^2}{2} \left(\sum_{i=1}^N Q_i |\phi_i|^2 - r \right)^2 + \sum_{i=1}^N Q_i^2 |\sigma|^2 |\phi_i|^2 \quad . \quad (2.9)$$

For all non-zero finite values of r , the classical vacuum equation $V = 0$ results in a Higgs branch of vacua corresponding to a non-compact Calabi-Yau manifold while the vector multiplet scalar is forced to vanish: $\sigma = 0$. However, for $r = 0$ the classical Coulomb branch opens up, parameterized by non-zero σ while the chiral multiplet scalars are now forced to vanish: $\phi_i = 0$.

To see whether the Coulomb branch survives in the quantum theory, we compute the exact potential energy in the large $|\sigma|$ region of the field space. All charged matter is heavy and may be safely integrated out, resulting in an effective twisted superpotential \tilde{W} [12, 3, 13]¹:

$$\begin{aligned} \tilde{W} &= -t\Sigma - \sum_{i=1}^N Q_i \Sigma (\log(Q_i \Sigma) - 1) \\ &= -\Sigma \left(t + \sum_{i=1}^N Q_i \log Q_i \right), \end{aligned} \quad (2.10)$$

¹Throughout the paper, we use the convention of [16, 11] in which the action is multiplied by $\frac{1}{2\pi}$ in the exponent of the path-integral weight.

where $\sum_i Q_i = 0$ is used in the last step. The lowest energy density is given by [3, 14]

$$U(\sigma) = \frac{e_{\text{eff}}(\sigma)^2}{2} \min_{n \in \mathbb{Z}} \left| t + \sum_{i=1}^N Q_i \log Q_i + 2\pi i n \right|^2, \quad (2.11)$$

where $e_{\text{eff}}(\sigma)$ is an effective gauge coupling that approaches the bare value e as $|\sigma| \rightarrow +\infty$. Thus the truly flat Coulomb branch appears when

$$t = - \sum_{i=1}^N Q_i \log Q_i \quad (\text{modulo } 2\pi i \mathbb{Z}). \quad (2.12)$$

The main quantum effect is the contribution of the 2π periodic theta angle θ ; a secondary effect is the finite shift of the parameters. The end result is that there is a *single* value of the complex FI parameter t for which the Coulomb branch emerges. At this point, correlation functions of Higgs branch operators diverge. For this reason the value of t (2.12) is referred to as a *singular* point in the one-dimensional quantum Kähler moduli space. In the dual mirror theory, it corresponds to the point of the complex structure moduli space where the mirror manifold develops a conifold singularity [15–17]. By abuse of language, we will also refer to the singular point in the Kähler moduli space as the “conifold point”.

In models with more than one Kähler moduli, the multi-dimensional FI-parameter space is separated into various “phases”. With the inclusion of the theta angles, the “phase boundaries” lift up to a subvariety of complex codimension one where the theory is singular. (See, for example, figure 1-3 of [13] for a graphical representation of this in a specific example). Each phase boundary has an asymptotic region where the unbroken gauge group is one $U(1)$ and there too the number of singular loci is *exactly one*.

2.3 Non-abelian theories

We would now like to repeat this analysis for non-Abelian gauge theories. As we shall see, quantum corrections to the classical singularity are much more pronounced. We consider a $U(k)$ gauge theory with N chiral multiplets Φ_i transforming in the fundamental representation \mathbf{k} , and a further S chiral multiplets P_α transforming in the \det^{-q_α} representation. Again we turn off the superpotential, $W = 0$, as it does not affect the singularity analysis. As discussed in section 2.2, the criterion for conformal invariance is $\sum_\alpha q_\alpha = N$. The classical potential is

$$V = \frac{1}{2e^2} \text{Tr} [\sigma, \sigma^\dagger]^2 + \frac{e^2}{2} \text{Tr} \left(\sum_{i=1}^N \phi_i \phi_i^\dagger - \sum_{\alpha=1}^S q_\alpha |p^\alpha|^2 \mathbf{1}_k - r \mathbf{1}_k \right)^2 + \frac{1}{2} \sum_{i=1}^N \phi_i^\dagger \{ \sigma^\dagger, \sigma \} \phi_i + \sum_{\alpha=1}^S q_\alpha^2 |\text{Tr} \sigma|^2 |p^\alpha|^2. \quad (2.13)$$

The classical vacuum equation $V = 0$ first of all requires that $[\sigma, \sigma^\dagger] = 0$, that is, σ is diagonalizable,

$$\sigma = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_k \end{pmatrix}. \quad (2.14)$$

The eigenvalues are generically all distinct and the $U(k)$ gauge group is broken to its maximal torus $U(1)^k$. However, as some of the eigenvalues coalesce there is an enhanced unbroken non-Abelian subgroup. The classical theory has different Coulomb branches for different values of r . For $r > 0$, the D-term equation requires that $\phi\phi^\dagger$ has maximal rank k , breaking the entire $U(k)$ gauge group: the Coulomb branch is completely lifted ($\sigma = 0$) and we are left only with the Higgs branch which is a smooth non-compact Calabi-Yau space \mathcal{M} of dimension $Nk + S - k^2$. In contrast, for $r < 0$, the p 's must have non-zero values which break only the central $U(1) \subset U(k)$ subgroup. The other scalars ϕ can have various ranks, from 0 to k , and the residual $SU(k)$ gauge symmetry is broken to the complementary subgroups. The classical vacuum manifold at $r < 0$ is thus a mixed Coulomb-Higgs branch: Coulomb branches of various dimensions, from 0 to $(k-1)$, sit over loci of the Higgs branch classified by the rank of ϕ . Finally, for $r = 0$ the full k -dimensional Coulomb branch emerges at $\phi = p = 0$.

Let us examine the existence of a genuine quantum Coulomb branch by computing the exact effective potential. We assume that σ is diagonalizable (2.14), and furthermore

$$\begin{aligned} \sigma_a &\neq \sigma_b \quad \text{if } a \neq b, \\ \sigma_a &\neq 0, \quad \forall a, \\ \sum_{a=1}^k \sigma_a &\neq 0 \end{aligned} \tag{2.15}$$

to suppress all non-trivial interactions. Then all the charged multiplets and W-bosons are heavy and can be integrated out. This results in the effective twisted superpotential,

$$\tilde{W} = -t \sum_{a=1}^k \Sigma_a - \sum_{a=1}^k N \Sigma_a (\log \Sigma_a - 1) + \sum_{\alpha=1}^S q_\alpha \left(\sum_{a=1}^k \Sigma_a \right) \left[\log \left(-q_\alpha \sum_{a=1}^k \Sigma_a \right) - 1 \right]. \tag{2.16}$$

Note that there is no contribution from the W-boson integrals [9]. Let us see whether there is a true flat direction within the validity range (2.15) of this superpotential. First we show that with a fixed trace

$$\sum_{a=1}^k \Sigma_a = \text{fixed} \tag{2.17}$$

the remaining $(k-1)$ relative modes are massive. To see this we add the Lagrange multiplier term

$$\Delta \tilde{W} = \lambda \left(\Sigma - \sum_{a=1}^k \Sigma_a \right) \tag{2.18}$$

and extremize $\tilde{W} + \Delta \tilde{W}$ with respect to λ as well as the independent Σ_a 's:

$$\frac{\partial(\tilde{W} + \Delta \tilde{W})}{\partial \Sigma_a} = -N \log \Sigma_a + \sum_{\alpha=1}^S q_\alpha \log(-q_\alpha \Sigma) - (t + \lambda) = 0. \tag{2.19}$$

For a fixed value of Σ , the components Σ_a are constrained to obey $\Sigma_a^N = \exp(\Xi)$ where the quantity $\Xi = \sum_{\alpha} q_\alpha \log(-q_\alpha \Sigma) - (t + \lambda)$ is independent of the gauge index a . The solutions

to equation (2.19) take the form,

$$\Sigma_a = \frac{\omega^{n_a}}{Z} \Sigma \tag{2.20}$$

where $\omega = e^{2\pi i/N}$ and $n_a \in \{0, 1, \dots, N-1\}$ are some choice of k integers and

$$Z = \sum_{a=1}^k \omega^{n_a}. \tag{2.21}$$

These solutions do not come in continuous families, but are isolated: this is the statement that the relative Σ_a 's are massive in the background of a fixed trace Σ . For the solution (2.20) to lie in the range of validity (2.15), we require that the k choices n_a are all distinct and that $Z = \sum_a \omega^{n_a}$ is non-vanishing. Note also that the uniform shift $n_a \rightarrow n_a + 1 \forall a$ results in $Z \rightarrow \omega Z$ and does not change the ratio Σ_a . Finally, a permutation of the n_a 's is a gauge transformation.

To summarize: the number of trustworthy solutions for Σ_a given by equation (2.20) is equal to the number of distinct choices of k integers n_a from $\{0, 1, \dots, N-1\}$, modulo shifts $n_a \rightarrow n_a + 1$, with the requirement that $Z = \sum \omega^{n_a} \neq 0$. Let us denote this number as $n(k, N)$. It will play an important role in the following section.

Finally, we turn to the original question: when does a Coulomb branch for Σ arise? We may substitute the solution (2.20) into the effective superpotential (2.16) to find,

$$\tilde{W} = \Sigma \left(-t + N \log Z + \sum_{\alpha=1}^S q_\alpha \log(-q_\alpha) \right) \tag{2.22}$$

So, as with the Abelian case, we find that the Coulomb branch exists when the FI-theta parameter takes the specific values,

$$\prod_{\alpha} q_\alpha^{q_\alpha} e^{-t} = \frac{(-1)^N}{Z^N} \tag{2.23}$$

Only a one-dimensional Coulomb branch emerges from each of these points, rather than a k -dimensional branch that one might expect from a $U(k)$ gauge theory. Each Coulomb branch is parameterized by the scalar $\Sigma = \sum_a \Sigma_a$ associated to the quotient group $U(1) = U(k)/SU(k)$. The different branches are characterized by the mass spectrum of the W-bosons, given by $\Sigma_a - \Sigma_b$, where Σ_a satisfy (2.20).

For our immediate purposes, the crucial point is that the value of t for which the one-dimensional Coulomb branch exists is not unique; it exists for every non-vanishing value of $Z = \sum \omega^{n_a}$ with distinct integers $n_a \in \{0, 1, \dots, N-1\}$. Thus, in general, there are multiple singular points in the middle of the Kähler moduli space.

We have not yet analyzed the quantum fate of the mixed Coulomb-Higgs branches that exist classically at any negative value of r . We will show in section 4.4 that these are lifted and do not introduce further singularities in the Kähler moduli space.

2.4 Comparison with mirror symmetry conjecture

In [1] Batyrev, Ciocan-Fontanine, Kim and van Straten constructed the mirror manifold Y for each Calabi-Yau 3-fold X listed in table 1. The solution to the Picard-Fuchs equation provides the classical Yukawa coupling $K_z^{(3)}$ of the mirror theory in terms of the single complex structure z . In this section we check that the discriminant locus of Y coincides with the quantum singularities (2.23) found above. We deal with each example in turn, writing down the Yukawa coupling computed in [1] before comparing to the quantum singularities:

- The simplest 3-fold $X_4 \subset G(2, 4)$ may also be written as $V_{2,4} \subset \mathbb{CP}^5$, the intersection of a quadric, which restricts to $G(2, 4) \subset \mathbb{CP}^5$, with a quartic. The mirror was previously computed in [18] and [19] (see example 6.4.2 of the former paper).

Since this example may be rewritten as an Abelian model, we do not expect the singularity to split. Indeed, the correlation functions of the mirror have a single pole at $z = 2^{-10}$ [1]. This is to be compared with equation (2.23) which yields singularities when $4^4 e^{-t} = 1/Z^4$. For this example, there is a unique choice: $Z = 1 + \omega$. (Of the other choices, $Z = 1 + \omega^2 = 0$ so is invalid, while $Z = 1 + \omega^3$ is related to the first choice by a shift $Z \rightarrow \omega^3 Z$). The location of the pole agrees with the mirror under the identification $z = -e^{-t}$.

- For the next example $X_{1,1,3} \subset G(2, 5)$, the computation of Batyrev et al. yields

$$K_z^{(3)} = \frac{15}{1 - 11(3^3 z) - (3^3 z)^2} \tag{2.24}$$

The Coulomb branch analysis (2.23) gives singularities when $3^3 e^{-t} = -1/Z^5$. This time there are two possible choices with $Z \neq 0$. They are $Z = 1 + \omega$ and $Z = 1 + \omega^2$, yielding singularities at $3^3 e^{-t} = -11/2 \pm 5\sqrt{5}/2$ in agreement with the mirror symmetry prediction (2.24). Note that this tells us we should identify the coordinates as $z = e^{-t}$.

- The mirror of the Calabi-Yau $X_{1,2,2} \subset G(2, 5)$ has Yukawa coupling

$$K_z^{(3)} = \frac{20}{1 - 11(2^4 z) - (2^4 z)^2} \tag{2.25}$$

Comparing the denominator to (2.24), we see that the position of the singularities remains the same up to the normalization of z . Indeed, we see the same behavior in our quantum theory where the normalization arises from $\prod_{\alpha} q_{\alpha}^{g_{\alpha}} = 2^4$. Once again find agreement if we identify $z = e^{-t}$.

- The mirror of the Calabi-Yau $X_{1,1,1,1,2} \subset G(2, 6)$ has Yukawa coupling

$$K_z^{(3)} = \frac{28}{((2^2 z) + 1)(27(2^2 z) - 1)} \tag{2.26}$$

From (2.23), the singularities occur at $(2^2) e^{-t} = 1/Z^6$. There are two possibilities: $Z = 1 + \omega$ and $Z = 1 + \omega^2$. This gives $Z^6 = -27$ and 1 respectively, resulting in agreement for $z = -e^{-t}$

- The mirror of the Calabi-Yau $X_{1,\dots,1} \subset G(2, 7)$ has Yukawa coupling

$$K_z^{(3)} = \frac{42 - 14z}{1 - 57z - 289z^2 + z^3} \tag{2.27}$$

The quantum singularities of (2.23) occur at $e^{-t} = -1/Z^7$, where there are now three choices: $Z = 1 + \omega$, $Z = 1 + \omega^2$ and $Z = 1 + \omega^3$. It is a simple matter to check numerically that these reproduce the three poles of the Yukawa coupling with $z = e^{-t}$.

- Finally, the mirror of the Calabi-Yau $X_{1,\dots,1} \subset G(3, 6)$ has Yukawa coupling²

$$K_z^{(3)} = \frac{42}{(1 - z)(1 - 64z)} \tag{2.28}$$

The singularities from (2.23) lie at $e^{-t} = 1/Z^6$. This time we have *three* choices for Z . They are $Z = 1 + \omega + \omega^2$, $Z = 1 + \omega + \omega^3$ and $Z = 1 + \omega + \omega^4$. They yield $Z^6 = 64, 1, 1$ respectively. Note that the presence of the two solutions with $Z = 1$ appears not to lead to a double pole in the Yukawa coupling. Once again, the dictionary between t and the complex structure z chosen in [1] requires $e^{-t} = z$ for agreement.

It is worth noting that for each example, the map takes the form $e^{-t} = (-1)^{k+N+1} z$. Note that z is chosen in [1] so that it approaches e^{-T} in the large volume limit, where T is the complexified Kähler class $T = \int_C (\omega - iB)$. The sign $(-1)^{N+k+1}$ shows that the linear sigma model theta angle is asymptotically related to the B-field by

$$\int_C B \simeq \theta + \pi(N + k + 1) \text{ modulo } 2\pi\mathbb{Z}. \tag{2.29}$$

It would be interesting to understand the origin of this shift. (See [13] for a discussion of this point in Abelian models.)

3. The Witten index for two dimensional SQCD

In this section we calculate the Witten index for $\mathcal{N} = (2, 2)$ $SU(k)$ gauge theory with N chiral multiplets in the fundamental representation \mathbf{k} . To our knowledge, a computation of the index has not previously appeared in the literature. Recall that for a $U(k)$ gauge group coupled to N fundamental chiral multiplets there are $\binom{N}{k}$ vacua. The question is: what happens when we decouple the overall $U(1)$?

Theories with extended supersymmetry typically have non-compact moduli spaces of vacua making the computation of the Witten index tricky at best, ill-defined at worst [4]. Our model is no exception. When the chiral multiplets have vanishing mass, the classical theory has both a non-compact Higgs branch of complex dimension $k(N - k) + 1$ and a

²The original formula of [1] contains a minus sign typographical error in the denominator. We thank Duco van Straten for confirmation of this.

non-compact Coulomb branch of dimension $k - 1$. In this section, we deform the theory by endowing the chiral multiplets with twisted masses [5]. As we shall see, generic masses lift both Higgs and Coulomb branches and render the Witten index well-defined. We now compute this index, which we call $w(k, N)$. We postpone the discussion of massless flavors to section 4.

3.1 A first attempt at counting

We start with a direct approach. This will give the right answer despite a number of shortcomings which we later remedy by taking a more oblique route. Let us give the twisted mass \tilde{m}_i to the i -th fundamental field Φ_i . The scalar potential for the $SU(k)$ theory is

$$V = \frac{1}{2e^2} \text{Tr} |[\sigma, \sigma^\dagger]|^2 + \frac{e^2}{2} \text{Tr} \left[\sum_{i=1}^N \left(\phi_i \phi_i^\dagger - \frac{1}{k} \text{Tr}(\phi_i \phi_i^\dagger) \mathbf{1}_k \right) \right]^2 + \frac{1}{2} \sum_{i=1}^N \phi_i^\dagger \{ \sigma^\dagger - \tilde{m}_i^*, \sigma - \tilde{m}_i \} \phi_i. \tag{3.1}$$

The vacuum equation $V = 0$ requires $[\sigma, \sigma^\dagger] = 0$, the D-term equation

$$\phi \phi^\dagger = \frac{1}{k} \text{Tr}(\phi \phi^\dagger) \mathbf{1}_k, \tag{3.2}$$

for the $k \times N$ matrix $\phi := (\phi_1, \dots, \phi_N)$, and the mass equations

$$(\sigma - \tilde{m}_i \mathbf{1}_k) \phi_i = (\sigma^\dagger - \tilde{m}_i^* \mathbf{1}_k) \phi_i = 0. \tag{3.3}$$

The D-term equation requires that ϕ is either zero or of rank k . We shall say that the masses \tilde{m}_i are *generic* if there is no Higgs branch. Under this definition, equal non-zero masses $\tilde{m}_1 = \dots = \tilde{m}_N =: \tilde{m}$ are generic. To see this note that $(\sigma - \tilde{m} \mathbf{1}_k)$ is always non-zero because σ is traceless. This means that the mass equation $(\sigma - \tilde{m} \mathbf{1}_k) \phi = 0$ requires the rank of ϕ to be less than k , and hence it must be zero. The masses are non-generic if and only if there are distinct i_1, \dots, i_k such that $\tilde{m}_{i_1} + \dots + \tilde{m}_{i_k} = 0$; in that case there is a Higgs branch at $\sigma = \text{diag}(\tilde{m}_{i_1}, \dots, \tilde{m}_{i_k})$.

For generic twisted masses, the only possible flat direction is the Coulomb branch. Let us compute the effective action in the weakly coupled regime. We assume that σ is diagonalizable (2.14), with $\sigma_1 + \dots + \sigma_k = 0$, and furthermore

$$\sigma_a \neq \sigma_b \text{ if } a \neq b, \tag{3.4}$$

to suppress the strong non-Abelian gauge interactions, and

$$\sigma_a \neq \tilde{m}_i, \quad \forall a, i, \tag{3.5}$$

so that there are no massless charged fields. Then we can integrate out all massive fields, leaving only the gauge multiplets for the unbroken $U(1)^k$, to obtain the effective superpotential:

$$\tilde{W} = - \sum_{i=1}^N \sum_{a=1}^k (\Sigma_a - \tilde{m}_i) \left(\log(\Sigma_a - \tilde{m}_i) - 1 \right). \tag{3.6}$$

For an $SU(k)$ theory, the Σ_a 's sum to zero, but can be treated independently provided we add the Lagrange multiplier term $-\lambda \sum_{a=1}^k \Sigma_a$. The vacuum equations are

$$\prod_{i=1}^N (\Sigma_a - \tilde{m}_i) = e^{-\lambda}, \quad a = 1, \dots, k, \quad (3.7)$$

$$\sum_{a=1}^k \Sigma_a = 0. \quad (3.8)$$

We first show that the Coulomb branch is lifted for generic masses, meaning that there are no continuous families of solutions to these equations. To see this, note that such any flat direction must allow a solution for all values of $e^{-\lambda}$. In particular there must be a solution with $e^{-\lambda} = 0$. Such a solution exists if, for each a , there is some i_a such that $\sigma_a = m_{i_a}$. The i_a 's can be taken distinct from one another as long as σ_a 's are generically distinct on the assumed family of solutions. Since σ is traceless we have $\sum_{a=1}^k \tilde{m}_{i_a} = 0$. But this violates our genericity condition on the masses. Thus, the Coulomb branch is lifted for generic masses. We would now like to count the number of solutions.

To simplify the analysis, we consider a particular example of generic masses — the case of equal masses, $\tilde{m}_1 = \dots = \tilde{m}_N =: \tilde{m}$. The equations are solved by

$$\Sigma_a - \tilde{m} = \omega^{n_a} e^{-\lambda/N}, \quad a = 1, \dots, k, \quad (3.9)$$

$$-k\tilde{m} = (\omega^{n_1} + \dots + \omega^{n_k}) e^{-\lambda/N}, \quad (3.10)$$

where ω is $e^{2\pi i/N}$ and n_a are integers defined modulo N . The second equation requires that $Z := \omega^{n_1} + \dots + \omega^{n_k}$ is non-zero, in which case we can write

$$\Sigma_a = \tilde{m} - \frac{\omega^{n_a}}{Z} k\tilde{m}. \quad (3.11)$$

These solutions have the property that they are unchanged by shifts $n_a \rightarrow n_a + 1 \forall a$. The condition (3.5) is always satisfied, while the other (3.4) is obeyed if and only if the n_a 's are all distinct. Moreover, a permutation of n_a 's is a gauge symmetry. Counting the number of such $\{n_a\}$'s, modulo gauge equivalence, is exactly the same combinatoric problem which we encountered in section 2.3, with the answer denoted as $n(k, N)$. This tempts us to claim that the Witten index is

$$w(k, N) = n(k, N). \quad (3.12)$$

The above analysis has a weakness: we neglected regions that violate (3.4) or (3.5) in which we do not have convenient weakly coupled variables. It is possible that we may have missed some ground states that are lurking in these regions. In fact, for generic masses, the vacuum equations (3.7)–(3.8) have no solution that violates (3.5) but obeys (3.4). In particular, the potential grows towards the loci $\sigma_a = \tilde{m}_i$, making it unlikely that a ground state is supported there. But we still have to worry about the loci $\sigma_a = \sigma_b$ with strong non-Abelian gauge interactions. In the next subsection, we will prove that there are no additional contributions to the Witten index and the result (3.12) is indeed correct.

3.2 The absence of contributions from strong coupling regimes

To aid our search for potential vacuum states, it will prove useful to embed the $SU(k)$ theory into a theory in which the index is determined decisively. A natural choice is the $U(k)$ gauge theory of the type discussed in section 2. We introduce N massless fundamental chiral multiplets Φ_i and N chiral multiplets P^i in the \det^{-1} representation. We endow the P_i fields with twisted masses \tilde{m}'_i . The scalar potential of this theory is

$$\begin{aligned}
 V = & \frac{1}{2e^2} \text{Tr} |[\sigma, \sigma^\dagger]|^2 + \frac{e^2}{2} \text{Tr} \left(\sum_{i=1}^N \left(\phi_i \phi_i^\dagger - |p_i|^2 \mathbf{1}_k \right) - r \mathbf{1}_k \right)^2 \\
 & + \frac{1}{2} \sum_{i=1}^N \phi_i^\dagger \{ \sigma^\dagger, \sigma \} \phi_i + \sum_{i=1}^N |\text{Tr}(\sigma) - \tilde{m}'_i|^2 |p^i|^2
 \end{aligned} \tag{3.13}$$

We will shortly see that in the regime $r \ll 0$, this reduces to the $SU(k)$ theory of interest. But first we examine the opposite limit with $r \gg 0$. Here the vacuum manifold is $\sigma = p_i = 0$, while the ϕ_i parameterize the Grassmannian $G(k, N)$. Undoubtedly the Witten index of this theory is equal to the Euler character [4], which is $\chi(G(N, k)) = \binom{N}{k} = N!/(N-k)!k!$.

Let us now find these vacua in the Coulomb branch analysis. Once again placing ourselves at the weakly coupled region

$$\begin{aligned}
 \sigma_a & \neq \sigma_b \quad \text{if } a \neq b; \\
 \sigma_a & \neq 0, \quad \forall a; \\
 \sum_{a=1}^k \sigma_a & \neq \tilde{m}'_i \quad \forall i,
 \end{aligned} \tag{3.14}$$

and integrating out the heavy fields, we obtain the twisted superpotential,

$$\tilde{W} = -t\Sigma - \sum_{a=1}^k N\Sigma_a [\log \Sigma_a - 1] + \sum_{i=1}^N (\Sigma - \tilde{m}'_i) [\log(\tilde{m}'_i - \Sigma) - 1] \tag{3.15}$$

where $\Sigma = \sum_{a=1}^k \Sigma_a$. The critical points of this potential lie at

$$\frac{\partial \tilde{W}}{\partial \Sigma_a} = 0 \quad \Rightarrow \quad \Sigma_a^N = e^{-t'} \prod_{i=1}^N (\Sigma - \tilde{m}'_i) \tag{3.16}$$

where $t' = t - N\pi i$. For non-zero twisted masses, we may find the full set of $\binom{N}{k}$ isolated vacua in a trustworthy regime. The solutions to (3.16) fall into two categories:

- The first class of solution to (3.16) is analogous to those found in section 2.3:

$$\Sigma_a = \frac{\omega^{n_a}}{Z} \Sigma \quad \text{where} \quad \Sigma^N = Z^N e^{-t'} \prod_{i=1}^N (\Sigma - \tilde{m}'_i) \tag{3.17}$$

with $Z = \sum_a \omega^{n_a}$. Once again, we must choose distinct integers $n_a \in \mathbb{Z}/N\mathbb{Z}$, modulo shifts $n_a \rightarrow n_a + 1$ and subject to the constraint $Z \neq 0$. There are $n(k, N)$ such

choices but, this time, each gives rise to N different vacua arising from solving the N^{th} order polynomial equation for Σ . Thus this class of solutions gives us $N n(k, N)$ vacua. Since dividing out by the shifts is compensated in the counting by multiplying by N , the number of vacua $N n(k, N)$ is simply equal to the number of distinct choices of n_a such that $Z \neq 0$.

- The second class of solutions to (2.20) arises from choosing integers n_a such that $Z = 0$. We set

$$\Sigma_a = \omega^{n_a} S \quad \text{where} \quad S^N = e^{-t} \prod_i \tilde{m}'_i \quad (3.18)$$

Now we have no reason to divide out by the shifts $n_a \rightarrow n_a + 1$. Thus the number of vacua in this class is the number of distinct choices of n_a such that $Z = 0$.

Between the two classes of solutions, the total number of vacua is the number of distinct choices of n_a , or $\binom{N}{k}$. Happily, we have found all vacua in the weakly coupled regime where $U(k) \rightarrow U(1)^k$.

So much for $r \gg 0$. Now let us look at the theory with $r \ll 0$. Classically we have N vacua labelled by $i = 1, \dots, N$ with,

$$\phi_j = 0, \quad |p^j|^2 = |r|\delta^{ij}, \quad \text{Tr}(\sigma) = \tilde{m}'_i \quad (3.19)$$

In each of these vacua the $U(k)$ gauge group is broken down to $SU(k)$ at the scale $|r|e^2$. Below this scale, each vacuum contains

$SU(k)$ gauge theory with N fundamental chiral multiplets.

In the i^{th} vacuum, all the fundamental chiral multiplets inherit a common twisted mass, $\tilde{m}_j = -\tilde{m}'_i/k$ ($\forall j$), and this sector has a well defined index $w(k, N)$. There are also $(N-1)$ singlet fields p^j with non-zero masses $\tilde{m}'_j - \tilde{m}'_i$, but they have one ground state each and do not change the index. Since we have N isolated classical vacua, the total number of vacua in this semi-classical analysis is $N w(k, N)$. But clearly this can't be all of them since $\binom{N}{k}$ is not always divisible by N . Where are the vacua that we've missed?!

To aid our search for the missing vacua, let's return to the Coulomb branch analysis. The semi-classical limit is trustworthy for $r \rightarrow -\infty$, and we may follow the fate of each vacuum in this limit. The solutions of the first type (3.17) survive with $\Sigma \rightarrow \tilde{m}'_i$. These are identified with the classical vacua (3.19). However, the solutions of the second type (3.18) go to infinity in the field space $\Sigma_a \rightarrow \infty$ in the classical limit $t' \rightarrow -\infty$; these are the vacua missed by the semi-classical discussion above. Since we found that the number of vacua of the first type is $N n(k, N)$, we conclude that the Witten index of $\mathcal{N} = (2, 2)$ $SU(k)$ SQCD with N chiral multiplets in the fundamental representation is indeed

$$w(k, N) = n(k, N). \quad (3.20)$$

The answer using this more careful method agrees with the direct counting in $SU(k)$ SQCD (3.12), proving that there are indeed no contributions from the strongly coupled

regime $\sigma_a = \sigma_b$. Note also that the result (3.20) holds for non-equal twisted masses, $\tilde{m}_i \neq \tilde{m}_j$, as long as they are generic.

We remind the reader that $n(k, N)$ arises from the combinatoric problem of picking k distinct mod- N integers $n_a \in \mathbb{Z}/N\mathbb{Z}$, modulo shifts $n_a \rightarrow n_a + 1$, such that $Z = \sum_a e^{2\pi i n_a/N} \neq 0$. Here we briefly sketch how one may perform this counting. Using the shift symmetry $n_a \rightarrow n_a + 1$, we may set $n_1 = 0$, leaving us to find a set of $(k - 1)$ distinct non-zero mod- N integers $\{n_2, \dots, n_k\}$. If the $Z \neq 0$ condition is ignored, there are $\binom{N-1}{k-1}$ such sets. However, these sets sit in families which must be identified by the shift symmetry. For example, the $(0, n_2, \dots, n_k)$ must be identified with $(-n_2, 0, n_3 - n_2, \dots, n_k - n_2)$ and so on. This means our sets of $(k - 1)$ integers are also identified: $\{n_2, \dots, n_k\}$, $\{-n_2, n_3 - n_2, \dots, n_k - n_2\}$, \dots , $\{-n_k, n_2 - n_k, \dots, n_{k-1} - n_k\}$ all lie in the same family and collectively count only one towards $w(k, N)$. These k sets are all different if the $Z = 1 + \omega^{n_2} + \dots + \omega^{n_k} \neq 0$ condition is met: For example, the replacement of the first set by the second one is achieved by the rescaling $\omega^{n_i} \rightarrow \omega^{n_i - n_2}$ but that would transform Z into $Z\omega^{-n_2}$ which is different from Z if $Z \neq 0$. Thus we find

$$n(k, N) = \frac{1}{k} \left\{ \binom{N-1}{k-1} - \# \right\} \tag{3.21}$$

where $\#$ is the number of $\{n_2, \dots, n_k\}$'s such that $Z = 0$.

Let us illustrate this counting for low rank gauge groups. For $SU(2)$ with N odd, there is no n_2 such that $1 + \omega^{n_2} = 0$, and therefore $w(2, N) = \frac{1}{2} \binom{N-1}{1} = \frac{(N-1)}{2}$. In contrast, if N is even, only $n_2 = \frac{N}{2}$ yields $1 + \omega^{n_2} = 0$. Thus $w(2, N) = \frac{1}{2} \{ \binom{N-1}{1} - 1 \} = \frac{(N-2)}{2}$. For $SU(3)$, $Z = 0$ is impossible if N is not divisible by 3, and thus $w(3, N) = \frac{1}{3} \binom{N-1}{2} = \frac{(N-1)(N-2)}{6}$ in that case. If N is a multiple of 3, only $\{n_2, n_3\} = \{\frac{N}{3}, \frac{2N}{3}\}$ yields $Z = 0$. Hence $w(3, N) = \frac{1}{3} \{ \binom{N-1}{2} - 1 \} = \frac{N(N-3)}{6}$. For higher k , there are more complex patterns with $Z = 0$.

The results for low values of k and N are shown in table 1 in the Introduction.

3.3 Varying the twisted masses

It is natural to claim that the index (3.12) truly counts the number of supersymmetric ground states (rather than merely counting ground states weighted with $(-1)^F$). This would mean that the only ground states lie in the weakly coupled regime and are solutions to (3.7)–(3.8). We can make some consistency checks of this proposal by following the fate of the ground states in various limits of the twisted masses.

$\tilde{m} \rightarrow \infty$. We claim that $SU(k)$ with $N \leq k$ massive flavors has no supersymmetric ground state, while the theory with $N = k + 1$ has a unique supersymmetric ground state. At first blush this is reminiscent of $\mathcal{N} = 1$ SQCD in four dimensions where the $SU(k)$ theory with N massless flavors has no supersymmetric ground state for $1 \leq N \leq k - 1$ [20]. However, there's an important difference. In the four dimensional theory, if the $N < k$ flavors have mass $m \gg \Lambda$ then supersymmetric vacua do exist, sitting at a distance $\sim 1/m$ in field space. As $m \rightarrow 0$, the vacua move to infinity, while as $m \rightarrow \infty$ they coalesce

around the strong coupling scale Λ . This behavior is necessary to accommodate the well-known fact that the Witten index for four dimensional pure $SU(k)$ theory is k . In contrast, the two dimensional theories have no vacua for $N \leq k$, even in the case of massive chiral multiplets. Relatedly, for $N > k$ multiplets with equal masses \tilde{m} , the vacua sit at a distance $\sim \tilde{m}$ in the field space. See Eqn (3.11). If we decouple the matter fields by sending the common twisted mass \tilde{m} to infinity, all these vacua disappear from the finite region in the field space. Although our calculation above was only valid for theories with $N > 0$ chiral multiplets, these facts strongly suggest that $2d$ $(2, 2)$ pure $SU(k)$ Yang-Mills theory has no supersymmetric ground state. Recent advances in constructing lattice models for supersymmetric gauge theories (see [21] for a review and references) suggest that it may be possible to test this claim numerically.

The situation also bears a resemblance to three-dimensional $\mathcal{N} = 2$ SQCD with *real masses* m_r which are the $3d$ analogs of the $2d$ twisted masses [22]. In that case, there is a moduli space of supersymmetric vacua consisting of branches located at distances $\sim m_r$ from the origin of the Coulomb branch [23, 24]. As $m_r \rightarrow \infty$, the supersymmetric branches move away to infinity and we are left with the Coulomb branch of pure Yang-Mills theory, which is lifted by a superpotential [25]. In contrast, in two-dimensions the superpotential vanishes in the $\tilde{m} \rightarrow \infty$ limit. Nevertheless, the claimed absence of vacua in the strongly coupled regime in $N > 0$ theories suggests that in two dimensions, as in three, there is no supersymmetric ground state in the pure $\mathcal{N} = (2, 2)$ Yang-Mills theory.

An example. For illustration let us consider the $SU(2)$ theory with $N = 5$ flavors and distinct twisted masses. We choose one chiral multiplet to have mass \tilde{m}_1 , while the four remaining multiplets have mass \tilde{m}_2 . The vacuum equations (3.7)–(3.8) yield the equation

$$(\Sigma - \tilde{m}_1)(\Sigma - \tilde{m}_2)^4 = (-\Sigma - \tilde{m}_1)(-\Sigma - \tilde{m}_2)^4, \tag{3.22}$$

for $\Sigma := \Sigma_1 = -\Sigma_2$, which has five solutions

$$\Sigma = 0, \pm i\tilde{m}_2, \pm \sqrt{-4\tilde{m}_2^2 - 5\tilde{m}_1\tilde{m}_2}. \tag{3.23}$$

The solution $\Sigma = 0$ is in the strong coupling regime $\Sigma_1 = \Sigma_2$ and must be omitted. Further, the solutions $\Sigma = +M$ and $\Sigma = -M$ should be identified since they are related by the permutation $\Sigma_1 \leftrightarrow \Sigma_2$. Thus there are in fact two vacua: one with $\Sigma = \pm i\tilde{m}_2$ and another with $\Sigma = \pm \sqrt{-4\tilde{m}_2^2 - 5\tilde{m}_1\tilde{m}_2}$. This reproduces the counting $w(2, 5) = 2$. We can now consider various limits. Sending $\tilde{m}_2 \rightarrow \infty$ decouples four of the matter fields and, from the index $w(2, 1) = 0$, we expect to find no surviving ground states. Indeed, it is simple to see that both vacua run to infinity. Alternatively, sending $\tilde{m}_1 \rightarrow \infty$ decouples only a single chiral multiplet. Now the vacuum at $\Sigma = \pm \sqrt{-4\tilde{m}_2^2 - 5\tilde{m}_1\tilde{m}_2}$ runs away, while the vacuum at $\Sigma = \pm i\tilde{m}_2$ remains. This is in agreement with the counting $w(2, 4) = 1$.

$\tilde{m} \rightarrow 0$. The claim that there are no supersymmetric ground states for $N \leq k$ massive flavors implies that there are also no normalizable supersymmetric ground states when the twisted masses are turned off. To see this, suppose that it was not true and a normalizable supersymmetric ground state appeared in the massless theory. Then turning on masses

would only improve the long-distance behavior of the wavefunction and the ground state would survive in the massive theory as well, in contradiction to our claim. We will revisit massless SQCD in section 4.

3.4 Complex masses for SU(2) theories

We have seen that as the twisted masses $\tilde{m} \rightarrow \infty$, they carry some vacua to infinity with them in the σ_a field space. In this manner, ground states decouple from the theory as the number of chiral multiplets decrease.

For SU(2) SQCD with N fundamentals, the theory admits another, complex, mass term. This is allowed because the baryons are quadratic and one may add a gauge invariant superpotential,

$$W = \sum_{i,j=1}^N m^{ij} \epsilon_{ab} \Phi_i^a \Phi_j^b. \tag{3.24}$$

(There is no analog of this in SU(k) SQCD with $k > 2$ since the gauge invariants have power k or higher in the fundamentals Φ_i .) We would like to understand the effect of these complex masses on the ground state spectrum. By continuity, any supersymmetric vacua that exist at $m^{ij} = 0$ must survive at finite m^{ij} , and the index remains $n(2, N) = \lfloor \frac{N-1}{2} \rfloor$. The question we want to ask is: what happens as some part of m^{ij} is sent to infinity? Suppose a rank $2l$ sub-matrix becomes infinite, decreasing the number of flavors from N to $N - 2l$. The number of vacua must correspondingly decrease by l . But how does this happen? By the decoupling of the chiral and the twisted chiral sectors [11], m^{ij} cannot enter into the twisted superpotential. This rules out the possibility that the locations of the vacua move to infinity in the σ_a space as we vary the masses m^{ij} . There must be a different mechanism at work.

The key to understanding the fate of the vacua is the derivation of the correct effective Lagrangian on the Coulomb branch in the presence of the complex mass. Let us study this in a simpler model: U(1) gauge theory with two fields Φ_1, Φ_2 of charge 1, -1 , each with a common twisted mass \tilde{m} , together with the complex mass term

$$W = m\Phi_1\Phi_2. \tag{3.25}$$

The effective Lagrangian is obtained by integrating out Φ_1 and Φ_2 . The bosonic determinants yield the potential term

$$\int \frac{d^2k}{(2\pi)^2} \left[\log(k^2 + |\sigma - \tilde{m}|^2 + |m|^2 + D) - \log(k^2 + |\sigma + \tilde{m}|^2 + |m|^2 - D) \right] \tag{3.26}$$

for a constant profile of σ and D . If $|m| \ll |\sigma \pm \tilde{m}|$, one may expand the log as

$$\log(k^2 + |\sigma \mp \tilde{m}|^2 + |m|^2 \pm D) = \log(k^2 + |\sigma \mp \tilde{m}|^2) + \frac{|m|^2 \pm D}{k^2 + |\sigma \mp \tilde{m}|^2} + \dots \tag{3.27}$$

The first term is cancelled by the fermionic determinants while the second term leads to a part of the twisted F-term associated with the twisted superpotential

$$\tilde{W} = -t\Sigma - (\Sigma - \tilde{m}_1) \left[\log(\Sigma - \tilde{m}_1) - 1 \right] + (\Sigma + \tilde{m}_2) \left[\log(-\Sigma - \tilde{m}_2) - 1 \right]. \tag{3.28}$$

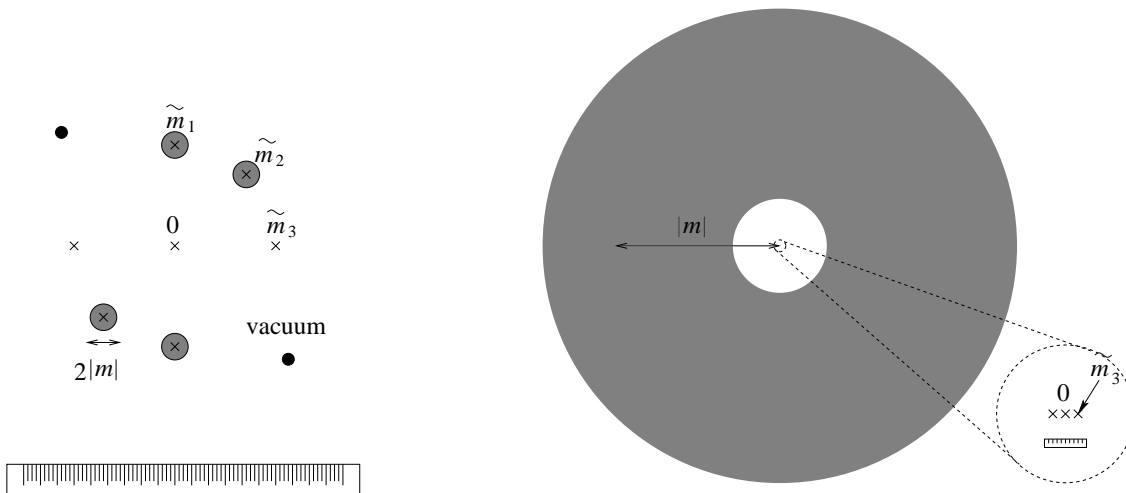


Figure 2: The Coulomb branch for SU(2) SQCD with $N = 3$. The scales of the left part ($|m| \ll |\tilde{m}_i|$) and the right part ($|m| \gg |\tilde{m}_i|$) are different.

We have employed twisted superpotentials of this type throughout this paper. On the other hand, in the opposite regime $|m| \gg |\sigma \pm \tilde{m}|$, the above expansion is not valid and it is more appropriate to write

$$\log(k^2 + |\sigma \mp \tilde{m}|^2 + |m|^2 \pm D) = \log(k^2 + |m|^2) + \frac{|\sigma \mp \tilde{m}|^2 \pm D}{k^2 + |m|^2} + \dots \quad (3.29)$$

We do not have the twisted superpotential (3.28) but some other effective Lagrangian in terms of the same field Σ . While we do not specify precisely this Lagrangian, it is sufficient to note that it can be expanded in powers of $(\sigma \pm \tilde{m})/m$ and vanishes in the limit $|m| \rightarrow \infty$.

We now apply this consideration to our SU(2) SQCD. Let us first consider a specific example: $N = 3$ flavors with generic twisted masses $\tilde{m}_1, \tilde{m}_2, \tilde{m}_3$, and with a complex mass m for the first and the second fields

$$W = m\epsilon_{ab}\Phi_1^a\Phi_2^b. \quad (3.30)$$

If $|m| \ll |\tilde{m}_i|$, the effective theory based on the superpotential (3.6) is valid except inside the two discs with radius $|m|$ and centers $\sigma_1 = \pm\tilde{m}_1, \pm\tilde{m}_2$. See the left part of figure 2.

The one vacuum state localized at the critical point of (3.6) is essentially unaffected by the presence of these discs. However, as $|m|$ is increased past $|\tilde{m}_i|$'s, the discs grow, merge into one domain and swallow the critical point. Inside that domain, the effective theory has the F-term potential associated with only the $i = 3$ part of the superpotential

$$\tilde{W}_3 = -(\Sigma_1 - \tilde{m}_3)(\log(\Sigma_1 - \tilde{m}_3) - 1) + (\Sigma_1 + \tilde{m}_3)(\log(-\Sigma_1 - \tilde{m}_3) - 1), \quad (3.31)$$

plus some additional Lagrangian \mathcal{L}_{12} . Even when $|m| \gg |\tilde{m}_i|$, the ground state must remain as long as $|m|$ is finite, since the Lagrangian based on the full superpotential (3.6) is still valid for $|\sigma| \gg |m|$, ensuring that no state has run away to infinity. The ground

state is therefore supported within the domain $|\sigma| < |m|$. Deep inside the domain, the additional Lagrangian \mathcal{L}_{12} is very small and the potential arising from \tilde{W}_3 dominates. But, as we have seen in sections 3.1 and 3.2, this potential does not have a zero and pushes the ground state wavefunction away from the center of the Coulomb branch. We conclude that the ground state can have support only in a halo region of radius $\sim |m|$, as depicted in the right part of figure 2. As $|m|$ is sent to infinity, the halo disappears from our sight. This is how the unique ground state of the $N = 3$ theory disappears as the complex mass is sent to infinity.

To end this discussion, let us add further flavors with twisted masses $\tilde{m}_4, \dots, \tilde{m}_N$, but without complex masses. If $|m| \ll |\tilde{m}_i|$, there are $\lfloor \frac{N-1}{2} \rfloor$ ground states localized at the critical points of (3.6). If $|m| \gg |\tilde{m}_i|$, there are as many ground state as before but they are no longer localized at the same critical points. The effective Lagrangian inside the domain of radius $|m|$ is the F-term associated with the $i = 3, \dots, N$ part of the superpotential, $\tilde{W}_{3\dots N}$, together with another Lagrangian arising from the expansion of the type (3.29). $\lfloor \frac{N-1}{2} \rfloor - 1$ states are localized at the critical points of $W_{3\dots N}$ but one state is not. As we will explain in more detail in section 4.2, the character of the F-term potential in the region $|\tilde{m}_i| \ll |\sigma| \ll |m|$ depends on whether N is even or odd. For N even, the F-term potential is nearly zero while, for N odd, it approaches a positive constant density $e^2\pi^2/2$ (see section 4.2). Therefore we conclude that, if N is odd, the positive potential forces the extra state to lie in a halo of radius $\sim |m|$. In contrast, if N is even, the state may have extended support throughout the region $|\tilde{m}_i| \ll |\sigma| < |m|$. As $|m|$ is sent to infinity, for odd N the extra state disappears as the halo moves away to infinity. For even N the state spreads, and become non-normalizable. In both cases, we are left with the $\lfloor \frac{N-1}{2} \rfloor - 1$ localized states.

4. Infra-red dynamics of $SU(k)$ gauge theories

In this section, we study $SU(k)$ gauge theories with N *massless* fundamental chiral multiplets. This class of theories includes massless SQCD, as well as theories with superpotentials. We expect that such a theory flows to a non-trivial superconformal field theory (SCFT) in the infra-red limit. However, the existence of the Coulomb branch presents a potential problem for the infra-red dynamics. The Coulomb branch provides a flat non-compact direction in field space into which low-energy states may spread. Typically, the existence of such a Coulomb branch is signalled by divergences in the correlation functions of Higgs branch operators. Such behavior is seen at the conifold points of $\mathcal{N} = (2, 2)$ Abelian theories [3, 6] and at the special point of $\mathcal{N} = (4, 4)$ SQED [6–8]. In this section, we will derive the criterion for the existence of a quantum Coulomb branch and analyze the associated singularity. We start with the computation of the central charge of the SCFT.

4.1 IR central charge

In an $\mathcal{N} = (2, 2)$ supersymmetric gauge theory with a simple gauge group, the axial $U(1)$ R-symmetry is anomaly free because $\text{Tr} F_{01} = 0$ for any representation of the group. If the superpotential is homogeneous and respects the vector $U(1)$ R-symmetry, the theory

is expected to flow to an $\mathcal{N} = (2, 2)$ SCFT with the left and right $U(1)$ current algebras inherited from the vector and axial R-symmetries.

In such a case, one can learn much about the CFT by studying the chiral ring [26] of the UV gauge theory. In particular, the $\overline{\mathcal{Q}}_+$ -chiral ring includes a copy of the $\mathcal{N} = 2$ superconformal algebra that descends to become part of the right-half of the chiral algebra in the infra-red theory [27]. It consists of the R-current j_- and its superpartners G_-, \overline{G}_-, T_- which combine into a super-R-current \mathcal{J} obeying $\overline{D}_+ \mathcal{J} = 0$. The central charge $\widehat{c} = c/3$ of the CFT can be read from the j_- - j_- operator product expansion

$$j_-(x)j_-(y) \sim -\frac{\widehat{c}}{(x^- - y^-)^2}. \tag{4.1}$$

This computation has been performed in several examples — Landau-Ginzburg models [27], $\mathcal{N} = (0, 2)$ Abelian linear sigma models (where only T_- - T_- OPE matters [28]), and $\mathcal{N} = (2, 2)$ Abelian linear sigma models [29]. However, provided the left and right R-symmetries are correctly identified in the UV, there is a quicker way to compute the central charge. The idea is to couple the vector $U(1)$ R-symmetry to a gauge field and look at the anomaly that arises for the axial $U(1)$ R-symmetry. The axial anomaly takes the form

$$\partial_+ j_- = a F_{+-}, \tag{4.2}$$

and $\partial_- j_+ = a F_{+-}$, where $F_{+-} = \partial_+ A_- - \partial_- A_+$ is the curvature of the $U(1)_V$ gauge field. Taking the variation of a correlator including (4.2) with respect to $A_+ \rightarrow A_+ + \delta A_+$ and setting $A = 0$, we find

$$\left\langle \partial_+ j_-(x) \frac{i}{\pi} \int \delta A_+(y) j_-(y) d^2 y O_1 \cdots O_s \right\rangle = -a \partial_- \delta A_+(x) \left\langle O_1 \cdots O_s \right\rangle. \tag{4.3}$$

It follows that the current-current product expansion is of the form (4.1) with

$$\widehat{c} = a. \tag{4.4}$$

Thus, the infra-red central charge can be read by computing the axial anomaly in the system where the vector R-symmetry is gauged. In a system with Dirac fermions $\psi_{i\pm}, \overline{\psi}_{i\pm}$ where $\psi_{i\pm}$ has $U(1)_V$ charge q_i and $U(1)_A$ charge ∓ 1 , the number a that measures the anomaly is minus the sum of the $U(1)_V$ charges;

$$a = \sum_{i: \text{Dirac fermion}} (-q_i). \tag{4.5}$$

Let us apply this technique to the $SU(k)$ gauge theory with N massless fundamental chiral multiplets and a superpotential which is homogeneous of degree d in the baryonic variables $B_{i_1 \dots i_k}$,

$$W = G_d(B) \tag{4.6}$$

Since the superpotential must have vector and axial R-charges 2 and 0, the R-charges of $B_{i_1 \dots i_k}$ are $2/d$ and 0, implying that the constituent fundamental chiral multiplets Φ_i^a have

R-charges $2/dk$ and 0. In particular, the fermionic components $\psi_{i\pm}^a$ have vector R-charge $(2/dk - 1)$ and axial R-charge ∓ 1 . The gaugino λ_{\pm} has vector and axial R-charges 1 and ∓ 1 respectively. Adding these gives the central charge in the infra-red limit,

$$\widehat{c} = Nk \times \left(-\frac{2}{dk} + 1 \right) + (k^2 - 1) \times (-1) = \frac{N(dk - 2)}{d} - (k^2 - 1). \quad (4.7)$$

As a check, let us consider the case $k = N = 1$: a theory with no gauge group $SU(1) = \{1\}$, one variable $B = \Phi$ and degree d superpotential $W = \Phi^d$. This is a Landau-Ginzburg (LG) model that flows to the $\mathcal{N} = (2, 2)$ minimal model of level $(d - 2)$ in the infra-red limit [30, 31]. The formula (4.7) yields $\widehat{c} = (d - 2)/d = 1 - 2/d$, which is indeed the central charge for the minimal model (see for example [32]).

4.2 The singularity criterion

Let us now examine the potential singularity of the infra-red theory. To be specific, we first consider $SU(k)$ SQCD, the theory without superpotential. As mentioned earlier, the classical theory has both a non-compact Higgs branch of complex dimension $k(N - k) + 1$ and a non-compact Coulomb branch of dimension $k - 1$. How do quantum effects change this story? The Higgs branch survives, and the sigma model on it flows to an interacting SCFT. It has an asymptotic region where the sigma model is weakly coupled, which implies that the central charge is the dimension $\widehat{c} = k(N - k) + 1$. In fact, in that region the R-symmetry cannot act non-trivially on the bosonic coordinate fields [6], which fixes the vector and axial R-charges of Φ_i^a to be zero. The general formula (4.5) indeed yields

$$\widehat{c} = Nk \times (-(-1)) + (k^2 - 1) \times (-1) = Nk - k^2 + 1. \quad (4.8)$$

This theory itself is singular in the sense that the target space is non-compact in the asymptotic large ϕ directions. But we would like to focus on another type of singularity of the Higgs branch theory — the additional non-compactness at $\phi = 0$ associated with the existence of Coulomb branch [6].

We may repeat the analysis of the effective superpotential (2.16) in our model, now with a Lagrange multiplier required to ensure that $\Sigma = \sum_{a=1}^k \Sigma_a = 0$. It is not hard to see, following the analysis of section 2.3, that most of the Coulomb branch is lifted by quantum effects. Coulomb branches of at most one dimension may survive, with

$$\Sigma_a = \omega^{n_a} e^{-\lambda/N} \quad (4.9)$$

where $\omega^N = 1$ and $n_a \in \{0, 1, \dots, N - 1\}$. This is a suitable $SU(k)$ configuration only if the k distinct integers n_a can be found such that $Z = \sum_a \omega^{n_a} = 0$. If no such integers n_a can be found, then the Coulomb branch is completely lifted by quantum effects.

For example, for $SU(2)$ gauge theory with N fundamental chiral multiplets, there exists a quantum Coulomb branch if and only if N is even. To see this explicitly, we write the effective superpotential on the Coulomb branch obtained by integrating out the N chiral multiplets:

$$\widetilde{W} = -N\Sigma_1(\log(\Sigma_1) - 1) + N\Sigma_1(\log(-\Sigma_1) - 1) = N\pi i\Sigma_1. \quad (4.10)$$

This shows that a theta angle $\theta = N\pi$ for the unbroken $U(1)$ subgroup is generated. The theta angle is identified as a background electric field [14] that carries the electro-static energy with density $e^2\theta^2/2$. If N is even, pair creations of the charge 1 particles will completely screen the background electric field: then the energy density is zero and the Coulomb branch remains as an exact flat direction. If N is odd, pair creations cannot completely screen it — the field with $\theta = \pi$ or $-\pi$ will always remain. The Coulomb branch is lifted. For $SU(3)$ gauge theory, there is one Coulomb branch if and only if N is divisible by 3, in the direction $(\Sigma_1, \Sigma_2, \Sigma_3) = (S, e^{2\pi i/3}S, e^{4\pi i/3}S)$. For $SU(k)$ with higher k , there can be multiple one-dimensional Coulomb branches. For example, for $SU(4)$ theory with $N = 8$ fundamentals, there are *two* Coulomb branches: one is in the direction $(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4) = (S, iS, -S, -iS)$ and another is along $(S, -S, e^{2\pi i/8}S, -e^{2\pi i/8}S)$.

To summarize, we propose a criterion for the existence of the one-dimensional Coulomb branches or, equivalently, the criterion for a singularity in the infra-red fixed point of the Higgs branch theory: *SU(k) SQCD with N massless flavors is singular at the origin $\phi = 0$ if and only if there are k distinct N-th roots of unity whose sum vanishes.* It would be an interesting problem to study such singular conformal field theories, particularly when there are multiple Coulomb branches.

This criterion is applicable also in theories with a non-trivial superpotential W , because the computation of the effective superpotential \tilde{W} for the twisted chiral superfield is not affected. If W removes the non-compact Higgs branch and if the criterion reveals no singularity, then we obtain a completely regular superconformal field theory in the infra-red limit. For example, let us consider the superpotential $W = G_d(B)$ which is a homogeneous polynomial of degree d in the baryons $B_{i_1 \dots i_k}$. Suppose $G_d(B)$ is generic so that the equation in the Plücker coordinates $G_d(B) = 0$ defines a smooth hypersurface of the Grassmannian $G(k, N)$. Then the F-term equation

$$\frac{\partial W}{\partial \Phi_j^a} = \sum_{i_1 < \dots < i_k} \frac{\partial G_d(B)}{\partial B_{i_1 \dots i_k}} \frac{\partial B_{i_1 \dots i_k}}{\partial \Phi_j^a} = 0 \tag{4.11}$$

together with the D-term equation (3.2) has no solution other than $\Phi_i^a = 0$. This ensures that the superpotential removes the non-compact Higgs branch. The assignment of non-zero R-charge to Φ_i^a is then justified and we can safely say that the IR fixed point has central charge $\hat{c} = N(dk - 2)/d - (k^2 - 1)$, as we have computed in (4.7). Let us denote this fixed point by $\mathcal{C}_{k,N}(G_d)$. We claim that *the superconformal field theory $\mathcal{C}_{k,N}(G_d)$ is completely non-singular if and only if there are no k distinct N-th roots of unity that sum to zero.*

4.3 $N = k + 1$: duality with free SCFT and LG models

Perhaps the most striking aspect of our singularity criterion is that for many values of (k, N) , the Coulomb branch is lifted and the theory is non-singular at the origin $\phi = 0$. For any k , the $SU(k)$ gauge theory with $N = k + 1$ fundamentals is such an example. We would like to comment here on some consequence of the regularity in this class of examples.

Let us first consider SQCD, the theory without a superpotential: $W = 0$. For $N = k+1$ there are no algebraic relations among the N baryonic operators

$$B^i := \epsilon^{ij_1 \dots j_k} B_{j_1 \dots j_k}, \quad i = 1, \dots, N. \tag{4.12}$$

These provide global coordinates of the Higgs branch which is therefore holomorphically isomorphic to \mathbb{C}^N . Geometrically, however, the classical Higgs branch is a cone with the metric

$$ds^2 = \frac{\|dB\|^2}{\|B\|^{2-2/k}} \tag{4.13}$$

To see this, we first note that the global symmetry group $U(N)$ transforms the baryons as $B^i \mapsto B^j (g^{-1})_j^i \det g$ and hence the Higgs branch is a cone over the sphere S^{2N-1} . The metric can therefore be written as $ds^2 = f \|dB\|^2$ for some function $f = f(\|B\|)$. To fix the function f , we use the fact that the Higgs branch of the $N = k$ theory embeds into that of the $N = k + 1$ theory, and is identified as, say, the complex line $B_2 = \dots = B_N = 0$. The $N = k$ Higgs branch is simply the cone \mathbb{C}/\mathbb{Z}_k with deficit angle $2(k-1)\pi/k$ and hence we find $f = 1/\|B\|^{2-2/k}$.

The classical metric (4.13) has a conical singularity at the origin $B = 0$, the unique enhanced $SU(k)$ symmetry point. This singularity comes from elimination of the $SU(k)$ gauge multiplets that becomes massless at this point. However, we have learnt that this must be modified by quantum effects. In the quantum theory, we have seen that the Coulomb branch is lifted by the electrostatic energy with density $\sim e^2 \pi^2/2$, ensuring that the $SU(k)$ gauge multiplet has a mass of order e . The theory at energies much smaller than e is described purely in terms of the baryons B^1, \dots, B^N as the independent variables, and no singularity is expected at the origin $B = 0$. We conclude that the low energy theory is just the sigma model on the quantum Higgs branch where the conical singularity at $B = 0$ is smoothed out. We expect that the sigma model metric simply flattens at lower energies and that the infra-red fixed point is the sigma model on the flat \mathbb{C}^N . In summary, *SU(k) SQCD with $N = k + 1$ massless flavors flows to a free superconformal field theory with $\hat{c} = N$, described by the N baryonic operators.*

This behavior is reminiscent of four dimensional $\mathcal{N} = 1$ $SU(N_c)$ SQCD with $N_f = N_c$ flavors [34], where the singularity of the classical moduli space of vacua is smoothed by instanton corrections. However, in the four-dimensional example, the complex structure is modified and the moduli space is moved away from the point of flavor symmetry enhancement. In contrast, in our two dimensional example, the point $B = 0$ remains in the moduli (or the target) space; the metric is merely smoothly rounded there.

Let us now turn to the theories with non-trivial superpotential, which we take to be a homogeneous polynomial of degree d in the baryonic coordinates,

$$W = G_d(B^1, \dots, B^N). \tag{4.14}$$

As in 2d SQCD, the theory at energies much smaller than e is described in terms of the gauge invariant composites B^1, \dots, B^N . We claim that *the theory is dual at energies $\ll e$*

to the Landau-Ginzburg model with N variables X^1, \dots, X^N and superpotential

$$W = G_d(X^1, \dots, X^N). \quad (4.15)$$

In particular, the two theories flow to the same infra-red fixed point. We recall that the fixed point of the $SU(k)$ theory, denoted as $\mathcal{C}_{N-1,N}(G_d)$, has central charge

$$\widehat{c} = Nk \times \left(1 - \frac{2}{dk}\right) - k^2 + 1 = k + 1 - N\frac{2}{d} = N \left(1 - \frac{2}{d}\right). \quad (4.16)$$

This matches the central charge for the IR fixed point of the Landau-Ginzburg model [30, 31]. For $d = 1$ and 2 , the formula yields $\widehat{c} = -N$ and 0 respectively, and the theories do not flow to non-trivial fixed points in the deep infra-red limit. However, the low energy duality with the Landau-Ginzburg model should still hold. We examine these cases now:

$d = 1$: the superpotential is linear in baryons, $W = a_1 B^1 + \dots + a_N B^N$. The supersymmetry condition in the dual Landau-Ginzburg model reads $\partial_i W = a_i = 0$ ($\forall i$) and has no solution. Hence, when $N = k + 1$, a linear superpotential induces supersymmetry breaking for the low energy theory on the Higgs branch! For the $SU(2)$ case ($N = 3$), the linear superpotential means that the fundamentals have complex masses (3.24), $m^{ij} = a_k \epsilon^{kij}$, and one can actually understand this supersymmetry breaking from our study in section 3.4. We have seen there that, with a generic twisted mass, there is one supersymmetric ground state supported in a wide halo region of radius $\sim \|a\|$ in the $SU(2)$ Coulomb branch. There is no supersymmetric ground state near the center where the Higgs branch is located. This conclusion must continue to hold when the twisted masses vanish, as we argued in section 3.3. Thus, this Higgs branch theory has no supersymmetric ground state.

$d = 2$: let us choose a superpotential that is non-degenerate, say $W = m_1 (B^1)^2 + \dots + m_N (B^N)^2$ with all $m_i \neq 0$. The dual Landau-Ginzburg model is a theory with a unique supersymmetric ground state which has mass gap $\min\{|m_i|\}$. We conclude that the $SU(k)$ theory with a non-degenerate quadratic superpotential for the baryons also has a unique supersymmetric ground state with a mass gap.

A variant of this example arises when the baryons are coupled to singlets in a quadratic superpotential. For example, let us consider the $SU(2)$ theory with $N = 3$ fundamentals Φ_1, Φ_2, Φ_3 and 3 singlets A_1, A_2, A_3 which are coupled through the superpotential

$$W = \epsilon^{ijk} A_i \epsilon_{ab} \Phi_j^a \Phi_k^b = A_1 B^1 + A_2 B^2 + A_3 B^3. \quad (4.17)$$

At energies much lower than e , this is dual to a Landau-Ginzburg model of six variables X^i, A_i with superpotential $W = A_1 X^1 + A_2 X^2 + A_3 X^3$. In particular, the theory has a unique supersymmetric ground state with a mass gap. This example will prove important in section 5.

For the range $3 \leq d \leq N$, we will provide an alternative derivation of the duality in section 4.5.

4.4 Interlude — The $U(k)$ linear sigma model revisited

Let us return to the $U(k)$ linear sigma models that we introduced in section 2. We saw that, for $r < 0$, there exist mixed Coulomb-Higgs branches (see the discussion below (2.14)) and we postponed the analysis of the fate of these branches until a suitable later time. That time is now.

For a negative FI parameter, the p^α fields have vevs and the low energy gauge group is the unbroken $SU(k)$. According to the criterion derived in section 4.2, this low energy theory is singular for certain (k, N) because of the existence of the quantum Coulomb branch. Does this mean that the $U(k)$ linear sigma model has an infra-red singularity for these (k, N) ? Among the models giving rise to Calabi-Yau three-folds (see table 2), those with $(k, N) = (2, 4), (2, 6)$ and $(3, 6)$ should be singular by our criterion. Yet the mirror analysis of [1] shows no sign of a singularity in the finite $r \ll 0$ region of the moduli space. Moreover, an explicit computation of correlation functions in the Higgs branch theory similarly shows no hint of singular behavior for $r \ll 0$ [33]. What is going on?

We claim in fact that the mixed Coulomb-Higgs branch is lifted and the $U(k)$ theory has no singularity for any (k, N) as long as $r \ll 0$ is finite. This is due to a residual, non-trivial effect, left behind by the Higgsed $U(1)$ sector. To confirm this, let us compute the effective superpotential on the $SU(k)$ Coulomb branch at finite $r \ll 0$. We give slowly varying, large and distinct, traceless background eigenvalues $\tilde{\Sigma}_a$ to the fieldstrength Σ ,

$$\sum_{a=1}^k \tilde{\Sigma}_a = 0. \tag{4.18}$$

These eigenvalues set an energy scale which we write M . The fundamental fields Φ_i and W -bosons all have mass of order M , and must be integrated out to obtain the effective theory below the scale M . This effective theory consists of the $U(1)^k$ vector multiplets and S chiral multiplets P^α that are charged only under the trace part $U(1)_0$ of $U(1)^k$. (There is also a twisted superpotential, a remnant of the Φ_i 's.) We choose the basis of the charge lattice of $U(1)_0$ so that P^α has charge q_α . The sector of $U(1)_0$ and the P^α fields is essentially that of the linear sigma model for the weighted projective space $\mathbb{W}\mathbb{P}^{q_1, \dots, q_s}$, with the effective FI-parameter $r_0(M) = -r \gg 0$ at the cut-off scale M . The FI parameter r_0 runs towards smaller values as the energy is decreased and eventually enters into the negative region. Then it is appropriate to integrate out the P^α fields, just as we do in the $\mathbb{C}\mathbb{P}^{N-1}$ model, and we obtain an effective action for Σ_0 with “coupling constants” $\tilde{\Sigma}_a$. Since it is the same thing as integrating out Φ_i 's and P^α 's at the same time, obviously the end result is exactly equal to (2.16) with the replacement

$$\Sigma_a = \tilde{\Sigma}_a - \Sigma_0/k. \tag{4.19}$$

The final step is to integrate out Σ_0 , i.e. extremize the superpotential with respect to Σ_0 . Solving for Σ_0 in terms of $\tilde{\Sigma}_a$'s gives

$$\Sigma_0 = f(e^{-t}, \tilde{\Sigma}). \tag{4.20}$$

One may say that the Higgsed U(1) sector has dynamically induced an effective twisted mass $\Sigma_0/k = f(e^{-t}, \tilde{\Sigma})/k$ for the fundamentals of the SU(k) theory. Plugging (4.20) back into the superpotential, we obtain a non-trivial superpotential for $\tilde{\Sigma}_a$'s. This lifts the SU(k) Coulomb branch at finite $r \ll 0$.

Since the last steps above are computationally involved, let us illustrate how this works in the case of SU(2). The twisted superpotential for Σ_0 and ‘‘coupling’’ $\tilde{\Sigma}_1 = -\tilde{\Sigma}_2 =: \tilde{\Sigma} - \Sigma_0/2$ is³

$$\begin{aligned} \tilde{W} = & t\Sigma_0 - N(\tilde{\Sigma} - \Sigma_0)(\log(\tilde{\Sigma} - \Sigma_0) - 1) - N(-\tilde{\Sigma})(\log(-\tilde{\Sigma}) - 1) \\ & + \sum_{\alpha=1}^S q_\alpha \Sigma_0 (\log(-q_\alpha \Sigma_0) - 1). \end{aligned} \tag{4.21}$$

The extremum equation $0 = \partial\tilde{W}/\partial\Sigma_0 = t + N \log(\tilde{\Sigma} - \Sigma_0) - \sum_{\alpha=1}^S q_\alpha \log(q_\alpha \Sigma_0)$ is solved by

$$\Sigma_0 = f(e^{-t})\tilde{\Sigma}, \tag{4.22}$$

where $f = f(e^{-t})$ solves the equation

$$\prod_{\alpha=1}^S q_\alpha^{q_\alpha} f^N = e^t (1 - f)^N. \tag{4.23}$$

If we now plug this solution back into the superpotential, we find

$$\tilde{W} = \left[-N \log(1 - f) + N \log(-1) \right] \tilde{\Sigma}. \tag{4.24}$$

An exact Coulomb branch exists only when the coefficient of $\tilde{\Sigma}$ vanishes (modulo $2\pi i\mathbb{Z}$): that is, when $(f - 1)^N = 1$. This is solved by $f = 1 + \omega$ where $\omega^N = 1$ or when t is given by

$$\prod_{\alpha=1}^S q_\alpha^{q_\alpha} (1 + \omega)^N = e^t (-1)^N, \tag{4.25}$$

for $\omega^N = 1$. This solution lies in the trustworthy regime only when $\omega \neq \pm 1$. But this is nothing other than the singular point (2.23); we have simply recovered the original Coulomb branch obtained in section 2. We find that the classical SU(2) Coulomb branch at $r \ll 0$ is indeed completely lifted, both for odd N (as we saw for pure SU(2) SQCD in section 4.2) and also for even N .

Despite the discussion above, we expect that in the strict limit $r \rightarrow -\infty$, where the Higgs mass $e\sqrt{-r}$ goes to infinity, the effect of the U(1)₀ vanishes. Indeed, this can be seen in our calculation since, in this limit, $f(e^{-t}, \tilde{\Sigma})$ vanishes (see, for example, (4.23) for the SU(2) case). The superpotential on the SU(k) Coulomb branch should converge to that of the pure SU(k) gauge theory with massless flavors. For the SU(2) example above,

³Instead of the symmetric but formal ‘‘embedding’’ $\Sigma_1 = \tilde{\Sigma} - \Sigma_0/2$, $\Sigma_2 = -\tilde{\Sigma} - \Sigma_0/2$, we work with a genuine embedding $\Sigma_1 = \tilde{\Sigma} - \Sigma_0$, $\Sigma_2 = -\tilde{\Sigma}$ of U(1)₀ into U(1)². This is to respect the integral structure of charge lattice which is important when a pair creation of quantized charges is involved in the discussion.

in the limit $f \rightarrow 0$ one indeed recovers (4.10) from (4.24). Thus, according to our result of section 4.2, a one-dimensional Coulomb branch develops at $t = -\infty$ when there are k distinct N -th roots of unity that sum to zero. For such (k, N) , we expect that the CFT at $t = -\infty$ is singular.

4.5 RG flow and duality

We may extend the discussion of linear sigma models to $U(k)$ theories in which the axial $U(1)$ R-symmetry is anomalous. For illustration, let us consider the model consisting of N fundamental chiral multiplets Φ_1, \dots, Φ_N and one field P in the \det^{-d} representation, with superpotential

$$W = PG_d(B), \tag{4.26}$$

where $G_d(B)$ is a degree d polynomial in the baryons $B_{i_1 \dots i_k}$. At $r \gg 0$ the theory reduces to the non-linear sigma model on the hypersurface $X_{k,N}(G_d)$ of the Grassmannian $G(k, N)$ defined by the equation $G_d = 0$. We assume the G_d is generic so that the hypersurface is non-singular. At $r \ll 0$, the field P acquires a non-zero value and breaks the gauge group $U(k)$ to the subgroup G of elements g such that $\det^d g = 1$, which includes $SU(k)$ as a subgroup of index d , $G/SU(k) \cong \mathbb{Z}_d$. In the strict $r \rightarrow -\infty$ limit, the theory has gauge group G coupled to $N = k + 1$ fundamentals, with the superpotential $W = G_d(B)$. This can also be regarded as the theory with gauge group $SU(k)$ which is further gauged by the \mathbb{Z}_d symmetry generated by

$$\gamma : \Phi_i^a \rightarrow e^{2\pi i/kd} \Phi_i^a. \tag{4.27}$$

It flows in the infra-red limit to a \mathbb{Z}_d orbifold of the SCFT $\mathcal{C}_{k,N}(G_d)$ that we introduced in section 4.2. The orbifold is unique since the cohomology $H^2(\mathbb{Z}_d, U(1))$ vanishes [35].

For $d < N$ the FI parameter runs from $r \gg 0$ to $r \ll 0$; for $d = N$ the FI-theta parameter $t = r - i\theta$ is an exactly marginal parameter of the IR fixed points; and for $d > N$ it runs from $r \ll 0$ to $r \gg 0$. The effective superpotential \widetilde{W} on the Coulomb branch can be computed and the vacua can be found, exactly as before. \widetilde{W} for a fixed trace $\sum_{a=1}^k \Sigma_a =: \Sigma$ has $n(k, N)$ critical points (2.20). For each of them, the vacuum equation for the trace is

$$\Sigma^N = e^{-t} Z^N (-d\Sigma)^d, \tag{4.28}$$

which has $|N - d|$ solutions at non-zero values and a degenerate solution at zero. These $|N - d| \times n(k, N)$ non-zero solutions correspond to massive vacua since \widetilde{W} is non-degenerate there. The degenerate solution at $\Sigma = 0$ corresponds to a superconformal field theory. These aspects are precisely as in the $k = 1$ Abelian theories [36].

To summarize, we learned from the $U(k)$ linear sigma model that:

- $d < N$: The non-linear sigma model on the Fano hypersurface $G_d = 0$ in the Grassmannian $G(k, N)$ flows in the infra-red to the \mathbb{Z}_d orbifold of $\mathcal{C}_{k,N}(G_d)$. It has also $(N - d)n(k, N)$ massive vacua. The IR fixed point is non-singular if and only if there are no k distinct N -th roots of unity that sum to zero.
- $d = N$: The Kähler moduli space of the Calabi-Yau hypersurface $G_d = 0$ in $G(k, N)$ has one large volume limit, $n(k, N)$ singular points, and one point that corresponds

to the \mathbb{Z}_d orbifold of $\mathcal{C}_{k,N}(G_N)$. This last point is also singular if there are k distinct N -th roots of unity that sum to zero. Otherwise, it is a regular theory and the point is analogous to the Gepner point for the Fermat quintic in \mathbb{CP}^4 .

- $d > N$: The \mathbb{Z}_d orbifold of $\mathcal{C}_{k,N}(G_d)$ has a deformation which drives the theory to non-linear sigma model on the hypersurface $G_d = 0$ of general type in $G(k, N)$. The deformed theory also has $(d - N)n(k, N)$ massive vacua. The UV theory is non-singular if and only if there are no k distinct N -th roots of unity that sum to zero.

The renormalization group flows and marginal deformations described above are consistent with the central charge. The one for $\mathcal{C}_{k,N}(G_d)/\mathbb{Z}_d$ is

$$\widehat{c} = Nk - k^2 + 1 - 2N/d = \dim X_{k,N}(G_d) + 2(1 - N/d), \quad (4.29)$$

where $X_{k,N}(G_d)$ is the hypersurface $G_d = 0$ in the Grassmannian $G(k, N)$. For $d < N$ and $d > N$, the central charge of the IR theory is smaller than that of the UV theory. For $d = N$, the central charge of the CFT $\mathcal{C}_{k,N}(G_d)/\mathbb{Z}_d$ is the same as the dimension of the Calabi-Yau $X_{k,N}(G_d)$.

We also learned the Witten index of the orbifold theory $\mathcal{C}_{k,N}(G_d)/\mathbb{Z}_d$: It is equal to the Witten index of the sigma model on $X_{k,N}(G_d)$ minus (*resp.* plus) the number of Coulomb branch vacua $|N - d|n(k, N)$ for $d \leq N$ (*resp.* $d > N$). Note that $n(k, N)$ is given by the formula (3.21) in which $\#$ can be set equal to zero if there are no k -distinct N -th roots of unity that sum up to zero. Thus, the index of the conformal field theory $\mathcal{C}_{k,N}(G_d)/\mathbb{Z}_d$ is, if it is non-singular,

$$\text{Tr} (-1)^F = \chi(X_{k,N}(G_d)) - \frac{(N - d)(N - 1)!}{k!(N - k)!}. \quad (4.30)$$

The Euler number $\chi(X_{k,N}(G_d))$ of the hypersurface can be computed by Schubert calculus on the Grassmannian.

Duality. We derive a duality of two dimensional SCFTs, analogous to Seiberg duality in four-dimensions [37]. This is based on the equivalence of the two Kähler manifolds

$$G(k, N) \cong G(N - k, N). \quad (4.31)$$

We consider two linear sigma models corresponding to the degree d hypersurfaces in these (equivalent) spaces: The first is the $U(k)$ gauge theory with N fundamentals Φ_1, \dots, Φ_N and one field P in the \det^{-d} representation. These fields are coupled via a superpotential $W = PG_d(B)$. The second has gauge group $U(N - k)$, N fundamental fields Φ^1, \dots, Φ^N and \det^{-d} field P' . The superpotential is now $W = P'G'_d(B')$ where $G'_d(B')$ is the same polynomial as $G_d(B)$, with the replacement

$$B_{i_1 \dots i_k} = \epsilon_{i_1 \dots i_N} (B')^{i_{k+1} \dots i_N} \quad (4.32)$$

These two theories are equivalent at $r \gg 0$ since they reduce to the same sigma model. In the opposite limit $r \rightarrow -\infty$, the $U(k)$ (*resp.* $U(N - k)$) theory flow to the \mathbb{Z}_d orbifold of

the conformal field theory $\mathcal{C}_{k,N}(G_d)$ (*resp.* $\mathcal{C}_{N-k,N}(G'_d)$). If $d = N$, the FI-theta parameter is an exactly marginal parameter of the IR fixed points of both theories. Therefore the equivalence at $r \gg 0$ means the equivalence of the theories at $r \rightarrow -\infty$. If instead $d < N$, the FI parameters in both theories flow from $r \gg 0$ to $r \ll 0$. Both theories have $(N - d)n(k, N)$ massive vacua on the Coulomb branch and one superconformal field theory. Thus, the equivalence at $r \gg 0$ yields the equivalence of the SCFTs. For both $d < N$ as well as $d = N$, we found the duality between the \mathbb{Z}_d orbifold of $\mathcal{C}_{k,N}(G_d)$ and the \mathbb{Z}_d orbifold of $\mathcal{C}_{N-k,N}(G'_d)$. Unfolding the \mathbb{Z}_d by the quantum symmetry [38], we obtain a duality between the conformal field theories

$$\mathcal{C}_{k,N}(G_d) \xleftrightarrow{\text{dual}} \mathcal{C}_{N-k,N}(G'_d). \tag{4.33}$$

Equation (4.7) confirms that these theories have the same central charge. Also, from the discussion in section 4.2, we see if one of these SCFTs is singular, then the other is too with the same number of one-dimensional Coulomb branches. If $d > N$, the relation of the $U(k)$ and $U(N - k)$ linear sigma models is not strong enough to prove the duality but certainly is consistent with it. To conclude, we proved the duality (4.33) for $d \leq N$ as long as $\hat{c} > 0$, and we conjecture it for $d > N$.

For the case $N = k + 1$, the dual conformal field theory $\mathcal{C}_{1,N}(G_d)$ is based on the trivial $SU(1)$ gauge group and is simply the Landau-Ginzburg model with superpotential $W = G_d(X)$. This provides the promised, alternative derivation of the duality of section 4.3 in the case $3 \leq d \leq N$.

It is possible that the conformal field theories $\mathcal{C}_{k,N}(G_d)$ are mostly new, but it is also possible that some of them are already known. The latter is the case for the $k = 1$ (and hence $N = k + 1$) theories: $\mathcal{C}_{1,N}(G_d)$ with Fermat G_d is the tensor product of minimal models. Possible candidates for the $k > 1$ theories may be found in Kazama-Suzuki models [39] — a class of conformal field theories that can be realized as gauged Wess-Zumino-Witten models and include the minimal models as the $SU(2)_{d-2} \text{ mod } U(1)$ examples. It would be an interesting problem to find whether there is indeed a correspondence with such known models, using our knowledge of $\mathcal{C}_{k,N}(G_d)$ such as the central charge, Witten index, chiral ring, existence of discrete symmetries, etc. In theories with mass gaps, it was shown in [9] that the relation (4.31) corresponds to the level-rank duality of a related Wess-Zumino-Witten models. It would be interesting to see whether the duality (4.33) of our “new” CFTs can be understood from an alternative point of view.

4.6 A test of the IR singularity

We have seen that the classical mixed Coulomb-Higgs branches that exist in $U(k)$ linear models for $r < 0$ are lifted for all finite r . Nonetheless, in the strict $r \rightarrow \infty$ limit, a one-dimensional Coulomb branch for the unbroken $SU(k)$ does indeed open up provided one can find k distinct N^{th} roots of unity that sum to zero. Examining our list of Calabi-Yau 3-folds, the simplest example where the IR theory is expected to exhibit a singularity is $X_4 \subset G(2, 4)$. This, and related theories, are the subject of this section. We will make an independent test of the singularity by examining a dual description which has nothing to do with the $SU(2)$ Coulomb branch.

The Grassmannian $G(2, 4)$ has the amusing property that it can be realized as a quadric hypersurface in \mathbb{CP}^5 . This means that the $U(2)$ theory with $N = 4$ fundamental chirals Φ_i and a single \det^{-4} chiral P has a dual Abelian description: a $U(1)$ gauge theory with six charge 1 fields which we arrange as a four-by-four antisymmetric matrix $X_{ij} = -X_{ji}$, $i, j = 1, 2, 3, 4$, and two fields P_1 and P_2 of charge -2 and -4 respectively. A superpotential $\mathcal{W} = PG_4(B)$ for the $U(2)$ model is reproduced by a superpotential for the $U(1)$ theory,

$$W = P_1 G_2(X) + P_2 G_4(X), \tag{4.34}$$

where

$$G_2(X) = X_{12}X_{34} - X_{13}X_{24} + X_{14}X_{23}, \tag{4.35}$$

and $G_4(X)$ is the same as $G_4(B)$ where X_{ij} is inserted in place of B_{ij} . At $r \gg 0$ the X fields span the projective space \mathbb{CP}^5 . The superpotential $G_2(X) = 0$ cuts out the Grassmannian $G(2, 4)$ in \mathbb{CP}^5 (this is the Plücker relation), while $G_4(X) = 0$ further selects the Calabi-Yau in $G(2, 4)$. Thus the low energy theory agrees with that of the $U(2)$ model at $r \gg 0$. By analytic continuation the duality must hold for all values of $t = r - i\theta$.

As discussed in the previous section, in the strict $t \rightarrow -\infty$ limit, the $U(2)$ model becomes the \mathbb{Z}_4 orbifold of the singular conformal field theory $\mathcal{C}_{2,4}(G_4)$ with central charge $\hat{c} = 3$. How can we see this singularity in the dual $U(1)$ description? At $r \ll 0$, the low-energy physics of the $U(1)$ theory is governed by a hybrid Landau-Ginzburg/sigma model on a vector bundle over the weighted projective space $\mathbb{WP}^{1,2}$ (a tear drop), with the base and the fiber spanned by (P_1, P_2) and X_{ij} respectively. The superpotential ensures that the fibre X -fields are massive except over the singular point $P_1 = 0$ of the base. As $r \rightarrow -\infty$, the Higgs mass blows up and the $U(1)$ vector multiplet decouples together with one linear combination of (P_1, P_2) . But, most importantly, the size of the tear drop blows up. This non-compactness in the $U(1)$ model is the sign of a the singularity in the $\mathcal{C}_{2,4}(G_4)$ conformal theory.

One can take a closer look at the $U(1)$ theory. It seems appropriate to focus on the neighborhood of the point $P_1 = 0$ of the tear drop, where P_2 decouples and P_1 remains. Then the theory is the \mathbb{Z}_4 Landau-Ginzburg orbifold with six fields $X_{ij} = -X_{ji}$ of charge 1 and one field P_1 of charge 2 with superpotential

$$W = G_4(X) + P_1 G_2(X). \tag{4.36}$$

The R-charges are $1/2$ for X 's and 1 for P_1 . The model has central charge $\hat{c} = \sum_i (1 - q_i) = 6 \times (1 - \frac{1}{2}) + 1 \times (1 - 1) = 3$. We claim that the IR limit of this LG-model is dual to the \mathbb{Z}_4 orbifold of $\mathcal{C}_{2,4}(G_4)$. The duality must also hold when the \mathbb{Z}_4 orbifold is undone in both sides. Note that X_{ij} are elementary fields in this dual theory, while the corresponding operators B_{ij} are composites in the $SU(2)$ theory. Further, the dual model exhibits no strong gauge interactions. These features are somewhat reminiscent of the $SU(N_c)$ SQCD with $N_f = N_c + 1$ flavors in four-dimensions [34]. This LG model is obviously singular because the field P_1 becomes massless at $X = 0$. This is the Higgs branch manifestation of the singularity of the strong $SU(2)$ dynamics.

We may also consider the conformal field theory $\mathcal{C}_{2,4}(G_d)$ for other values of d . For $1 < d \leq 4$, employing the $U(2)$ and $U(1)$ linear sigma models as above, we find that this is dual to the IR fixed point of the Landau-Ginzburg model of seven variables, $X_{ij} = -X_{ji}$ and P_1 , with superpotential

$$W = G_d(X) + P_1 G_2(X), \tag{4.37}$$

where $G_2(X)$ is the quadratic polynomial (4.35). Both theories have central charge $\hat{c} = 5 - 8/d$. (For $d > 4$, the linear sigma models are not strong enough to prove the duality to (4.37) but are certainly consistent with it.) We again find that the dual theory is singular at $X = 0$.

5. The glop transition

In the previous section, we have shown that the finite $r \ll 0$ phase of the $U(k)$ linear sigma-models is non-singular. But we have yet to understand the nature of the low-energy physics at $r \ll 0$ (with the exception of the models dual to Abelian theories such as $X_4 \subset G(2, 4)$). In general [3], this regime is described by a gauged Landau-Ginzburg model fibered over a weighted projective space $\mathbb{WP}^{q_1, \dots, q_s}$. These models are typically rather tricky to study directly. However, one may hope that we could bring the knowledge learnt in this paper to bear on the problem. It turns out that we have indeed learnt enough to study the $r \ll 0$ phase of one further theory associated to the Calabi-Yau 3-folds listed in table 2. This is the theory associated to $X_{1, \dots, 1} \subset G(2, 7)$ and it is the subject of this section.

5.1 Rødland's conjecture

In [2] Rødland made a conjecture regarding the one-dimensional Kähler moduli space of a particular (2,2) superconformal field theory in two-dimensions. He claimed the moduli space has *two* large volume limits for Calabi-Yau target spaces X and Y , defined as follows:

- $X = X_{1, \dots, 1} \subset G(2, 7)$. This is a complete intersection Calabi-Yau in a Grassmannian $G(2, 7)$ defined by 7 generic linear equations of the Plücker coordinates. We have already met this object in section 2.4 where we computed the location of the three singularities in the interior of the moduli space.
- $Y = (\text{Pfaffian Variety in } \mathbb{CP}^{20}) \cap \mathbb{CP}^6$. The Pfaffian variety $\text{Pf}(\wedge^2 \mathbb{C}^7)$ in $\mathbb{CP}^{20} \cong \mathbb{P}(\wedge^2 \mathbb{C}^7)$ is defined as the locus of lines $A \in \wedge^2 \mathbb{C}^7$ such that $A \wedge A \wedge A = 0$ in $\wedge^6 \mathbb{C}^7$. This means that if we view A as a 7×7 antisymmetric matrix, then A lies within the Pfaffian variety if each 6×6 sub-matrix has zero Pfaffian (i.e. zero determinant). In other words, $A \in \text{Pf}(\wedge^2 \mathbb{C}^7)$ if $\text{rank}(A)$ is less than the maximal 6 which, since A is antisymmetric, means $\text{rank}(A) \leq 4$. Finally Y is defined as the intersection of $\text{Pf}(\wedge^2 \mathbb{C}^7)$ with a generic 6-plane \mathbb{CP}^6 in \mathbb{CP}^{20} .

Both X and Y have the same Hodge numbers $h^{2,1} = 50$ and $h^{1,1} = 1$ and, in particular, both have a one-dimensional Kähler moduli space. Rødland conjectures that they lie on

the same moduli space [2]. As evidence, he studied the Picard-Fuchs equation for the proposed mirrors of X and Y and found that the equations are the same. Importantly, X and Y are *not* birationally equivalent. This distinguishes this system from the familiar flop transition. Because of this important difference, we term the topology changing transition between X and Y the Grassmannian flop, or glop transition. We shall now study the glop transition from the perspective of the linear sigma model.

5.2 The genericity condition

The linear sigma model for $X = X_{1,\dots,1} \subset G(2, 7)$ is a $U(2)$ gauge theory with 7 chiral multiplets Φ_i transforming in the fundamental representation, and a further 7 chiral multiplets P^i transforming in the \det^{-1} representation. The restriction to the compact Calabi-Yau X is achieved through the introduction of a superpotential

$$W = \frac{\mu}{2} A_i^{jk} P^i \epsilon_{ab} \Phi_j^a \Phi_k^b \tag{5.1}$$

where μ is a mass scale and A_i^{jk} are coefficients that are anti-symmetric in the upper indices: $A_i^{jk} = -A_i^{kj}$. The F-term equations read

$$A_k^{ij} \epsilon_{ab} \phi_i^a \phi_j^b = 0, \quad p^k A_k^{ij} \phi_j^b = 0. \tag{5.2}$$

For $r \gg 0$, the D-term equation $D_b^a = 0$ (see (2.5)) requires that (ϕ_i^a) has rank two. If the second F-term condition (5.2) is strong enough to require $p^i = 0$ then, at energy scales below μ , the theory will flow to the non-linear sigma-model with target space X , defined by

$$X = \left\{ [\phi_i^a] \in G(2, 7) \mid A_k^{ij} \phi_i^1 \phi_j^2 = 0, k = 1, \dots, 7 \right\}. \tag{5.3}$$

However, we must ensure that the F-term condition (5.2) indeed requires $p^i = 0$ for all $[\phi_i^a] \in G(2, 7)$. This condition holds when the two 7×7 matrices $A_k^i(\phi^a) := A_k^{ij} \phi_j^a$ with $a = 1, 2$ have a rank 7 linear combination. This genericity condition on the coefficients A_i^{jk} is equivalent to the requirement that X is smooth since, for any $[\phi_i^a] \in G(2, 7)$ obeying (5.2), we can find 7 normal directions $\delta_l X$ such that $\delta_l(A_k^{ij} \phi_i^1 \phi_j^2) = A_k^{ij} ((\delta_l \phi_i^1) \phi_j^2 + \phi_i^1 \delta_l \phi_j^2)$, regarded as a 7×7 matrix (for indices l, k), is of rank 7. From now on we pick coefficients A_i^{jk} satisfying this genericity condition.

5.3 The physics at $r \ll 0$

Let us now turn to the other regime $r \ll 0$. In this case the D-term condition $D_b^a = 0$ requires some $p^i \neq 0$, so that p^i provide homogeneous coordinates on \mathbb{CP}^6 . We define the 7×7 anti-symmetric matrix

$$A(p)^{ij} := p^k A_k^{ij}. \tag{5.4}$$

Since $p^i \neq 0$ breaks the $U(2)$ gauge symmetry to $SU(2)$, for each point $[p^i] \in \mathbb{CP}^6$ the low energy theory consists of an $SU(2)$ gauge theory with 7 fundamental chiral multiplets ϕ_i^a , with a complex mass term which varies over the \mathbb{CP}^6 ,

$$W = \mu A(p)^{ij} \phi_i^1 \phi_j^2 \tag{5.5}$$

To determine the infra-red physics, we can work in a Born-Oppenheimer approximation, first examining the gauge theory degrees of freedom and subsequently treating the low-energy \mathbb{CP}^6 sigma-model fields. The character of the gauge theory over each point $[p^i] \in \mathbb{CP}^6$ depends on the rank of the mass matrix $A(p)^{ij}$,

$$R := \text{rank } A(p). \tag{5.6}$$

We have an $SU(2)$ gauge theory with R massive chiral multiplets and $(7 - R)$ massless chiral multiplets. An arbitrary anti-symmetric 7×7 matrix can have rank 0, 2, 4 and 6. However, the genericity conditions on A_i^{jk} described above mean that only $R = 4$ and $R = 6$ are possibilities. To see this, pick an orthonormal basis $\{\phi^1, \dots, \phi^7\}$ of \mathbb{C}^7 satisfying $\sum_{i=1}^7 \phi_i^\mu \phi_i^{\dagger\nu} = r\delta^{\mu\nu}$, where $\mu, \nu = 1, \dots, 7$. We can split this into three mutually orthogonal pairs $\{\phi^1, \phi^2\}$, $\{\phi^3, \phi^4\}$, $\{\phi^5, \phi^6\}$, together with one remaining orthogonal element ϕ^7 . Each pair defines a point in the Grassmannian $G(2, 7)$. The genericity condition requires that for each such pair $\{\phi^{2m-1}, \phi^{2m}\}$ there exists a rank 7 linear combination of $A_k^i(\phi^{2m-1})$ and $A_k^i(\phi^{2m})$. But, in turn, this ensures that the same linear combination of ϕ_i^{2m-1} and ϕ_i^{2m} is non-vanishing when acted upon by $A(p)^{jk}$ for any $[p] \in \mathbb{CP}^6$. By construction, we have three such mutually orthogonal vectors. The remaining vector ϕ^7 allows us to construct one more independent pair that is not annihilated by $A(p)$. We conclude that the genericity condition requires that the rank of $A(p)$ is at least 4 at any point $[p]$ of \mathbb{CP}^6 . This leaves us with $R = 4$ and $R = 6$. Let us first look at these in turn, starting with the $R = 6$ domain.

We work in the Born-Oppenheimer approximation, in which the p^i fields are first taken to be fixed. At a point $[p] \in \mathbb{CP}^6$ with $\text{rank } A(p) = 6$, the theory of fast variables in the Born-Oppenheimer approximation is the $SU(2)$ gauge theory with a single massless chiral multiplet and 6 chiral multiplets with complex masses of order $\mu|r|$. From the discussion of section 3.4, we find that this theory has 3 vacuum states supported in a halo of radius $\sim \mu|r|$ on the $SU(2)$ Coulomb branch. In the limit $\mu \rightarrow \infty$, in which the F-terms (5.2) restrict us to the manifold X in the regime $r \gg 0$, these vacua disappear from sight and decouple from the theory. We lose the supersymmetric ground states if $[p]$ is deep in the $R = 6$ domain. What happens as $A(p)$ degenerates to rank 4? On the rank $R = 4$ locus, we have an $SU(2)$ gauge theory with 3 massless chiral multiplets and 4 massive chiral multiplets. Again, from section 3.4, we know that there are 3 ground states, two of which are supported in a halo of radius $\sim \mu|r|$, while the remaining ground state is localized at the origin of the Coulomb branch. Hence, as $[p]$ approaches the $R = 4$ locus, one of the three ground states descends from the $\mu|r|$ halo, to lie in the center of the $SU(2)$ Coulomb branch. We thus conclude that the low energy theory is localized on the rank $A(p) = 4$ locus in \mathbb{CP}^6 .

We may now consider a second stage Born-Oppenheimer approximation, where we fix a slowly varying profile p_* within the rank four locus and quantize everything else. Here, “everything else” consists of the $SU(2)$ gauge multiplet, the seven fundamentals Φ_i^a and the modes of P^i in \mathbb{CP}^6 that are transverse to the $R = 4$ locus. Since the $R = 4$ locus has dimension three, there are $6 - 3 = 3$ transverse modes, which we denote by $\delta_j P$, $j = 1, 2, 3$. We may discard the four fundamental chirals that are massive due to the complex mass matrix $A(p_*)^{ij}$. We relabel the flavor index so that the first three fundamentals Φ_1, Φ_2, Φ_3

are massless. The chiral multiplets are thus coupled through the superpotential

$$W_{\text{fast}} = \mu \sum_{i,j=1}^3 A(\delta P)^{ij} \Phi_i^1 \Phi_j^2. \tag{5.7}$$

Because the rank four locus is defined by $\delta_1 p = \delta_2 p = \delta_3 p = 0$, we may assume the form

$$A(\delta P) = \begin{pmatrix} 0 & \delta_3 P & \delta_2 P \\ -\delta_3 P & 0 & \delta_1 P \\ -\delta_2 P & -\delta_1 P & 0 \end{pmatrix}. \tag{5.8}$$

This theory of fast variables is precisely the theory (4.17) which we considered in section 4.3. As we concluded there, it has a unique supersymmetric ground state with a mass gap. This mass gap validates the second stage Born-Oppenheimer approximation.

Thus the true low energy degrees of freedom correspond to the motion of p_* only. We conclude that the low energy theory at $r \ll 0$ is the non-linear sigma model whose target space is the rank $A(p) = 4$ locus in $\mathbb{C}\mathbb{P}^6$, that is, the Pfaffian variety Y . The Pfaffian Y is a Calabi-Yau threefold and hence the resulting conformal field theory has $\widehat{c} = 3$ as expected. In this manner, the strongly coupled non-Abelian gauge theory has two phases in which the low-energy physics is described by a sigma-model on inequivalent Calabi-Yau manifolds, X and Y . This confirms the conjecture of Rødland [2]. It is also notable that we have constructed the linear sigma model for a Calabi-Yau manifold which is *not* a complete intersection of hypersurfaces in a toric variety. To our knowledge, this is the first such construction.

5.4 A comment on D-brane categories

One consequence of the fact that X and Y sit in a common Kähler moduli space is that the categories of topological B-branes for sigma models on X and Y must be equivalent. In mathematical terms, X and Y must be derived equivalent, that is, the associated derived categories of coherent sheaves must be equivalent, $D^b(\text{Coh}(X)) \cong D^b(\text{Coh}(Y))$.⁴ This is particularly striking since X and Y are not birationally equivalent. A mathematical proof of the derived equivalence has been given by A. Kuznetsov in [40] as an application of his Homological Projective Duality [41]. The proof is also given independently in [42]. It would be very interesting to construct equivalences of categories from the physics point of view, as is done for Abelian theories [43].

5.5 Generalization

There exist straightforward generalizations of the above story. For example, we may replace 7 by N . Namely, consider a $U(2)$ linear sigma model with N fundamentals Φ_i^a and N fields P^j in the inverse determinant representation, $i, j = 1, \dots, N$ with the superpotential (5.1) in which i, j, k run over $1, \dots, N$. We require the same genericity condition on the coefficients A_i^{jk} : if ϕ^1 and ϕ^2 are linearly independent vectors in \mathbb{C}^N , the two $N \times N$

⁴This consequence was first brought to our attention by Edward Witten (April, 2005).

matrices $A^i_k(\phi^a) := A_k^{ij} \phi_j^a$ with $a = 1, 2$ have a rank N linear combination. Under this condition, the theory at $r \gg 0$ reduces to the non-linear sigma model on a smooth Calabi-Yau manifold $X_{[N]}$ in $G(2, N)$ defined by N linear equations for the Plücker coordinates. The dimension of $X_{[N]}$ is $d = 2N - 4 - N = N - 4$, which is the central charge of the infra-red SCFT,

$$\hat{c} = N - 4. \tag{5.9}$$

At $r \ll 0$, the p 's acquire non-zero values, spanning $\mathbb{C}\mathbb{P}^{N-1}$ and breaking $U(2)$ to $SU(2)$. To study the low energy physics, we may again work in the Born-Oppenheimer approximation, where the $SU(2)$ dynamics is classified by the rank R of the mass matrix $A(p)$. As in the $N = 7$ case, the genericity condition imposes a bound on the rank: For odd N we have $R \geq (N + 1)/2$, while for even N we have $R \geq N/2$. We list below the allowed ranks for low values of N :

$$\begin{array}{ll}
 N = 5 & R = 4 \\
 N = 7 & R = 6, 4 \\
 N = 9 & R = 8, 6 \\
 N = 11 & R = 10, 8, 6 \\
 N = 13 & R = 12, 10, 8 \\
 \dots & \dots
 \end{array}
 \qquad
 \begin{array}{ll}
 N = 6 & R = 6, 4 \\
 N = 8 & R = 8, 6, 4 \\
 N = 10 & R = 10, 8, 6 \\
 N = 12 & R = 12, 10, 8, 6 \\
 N = 14 & R = 14, 12, 10, 8 \\
 \dots & \dots
 \end{array}
 \tag{5.10}$$

Let us discuss the odd N and even N cases separately.

Odd N . For this case, the Born-Oppenheimer approximation tells us that the theory localizes on the locus $R \leq N - 3$ within $\mathbb{C}\mathbb{P}^{N-1}$, provided this locus is non-empty. This defines the Pfaffian variety $Y_{[N]}$ which has dimension $(N - 1) - 3 = N - 4$.

The case of $N = 9$ is entirely equivalent to $N = 7$, since for both these examples the only allowed non-maximal rank is $R = N - 3$. The second stage Born-Oppenheimer approximation for the $N = 9$ theory again shows that all degrees of freedom, other than the motion in the $R = 6$ locus, are massive. We conclude that the low energy theory at $r \ll 0$ is the sigma model on the Pfaffian $Y_{[9]}$. Thus, we again have the glop transition: There is a complex one dimensional Kähler moduli space with two large volume limits; one corresponds to a Calabi-Yau fivefold in the Grassmannian $G(2, 9)$ and the other to a Pfaffian Calabi-Yau $\text{Pf}(\wedge^2 \mathbb{C}^9) \cap \mathbb{C}\mathbb{P}^8$. There are four singular points in between.

For $N \geq 11$, there are sub-loci with lower ranks, $R \leq N - 5$, which correspond to the singularities in $Y_{[N]}$. Along such loci, more than three $SU(2)$ fundamentals become massless and we expect additional low energy degrees of freedom. The low energy theory is not a simple non-linear sigma model, which is anyway ill-defined, but a theory with extra degrees of freedom along the $R \leq N - 5$ loci. These extra degrees of freedom must be responsible for the “quantum resolution” of the singularity of $Y_{[N]}$.⁵

The case $N = 5$ is special in that there is no point with $R = N - 3$. Within the Born-Oppenheimer approximation, we only find two vacuum states supported in a region

⁵Alexander Kuznetsov pointed out that the Pfaffian Calabi-Yau $Y_{[N]}$ are singular, with the exception of $N = 7, 9$, and suggested that one should consider certain non-commutative resolution of this singular variety when the derived category is discussed [40].

away from the origin in the $SU(2)$ Coulomb branch. In particular, we cannot identify the low energy theory that may flow to a $\widehat{c} = 5 - 4 = 1$ SCFT which we do expect for any value of $t = r - i\theta$. This means that the Born-Oppenheimer approximation is not valid in this case.

Even N . We first note, following the discussion of section 3.4, that the “decoupling” of flavors with large complex masses is qualitatively different from the odd N case. The states that decouple are not supported away from the origin, but rather spread over a wide region in the $SU(2)$ Coulomb branch. This leads to the singularity which we expect in the strict $t = -\infty$ limit. Thus, we do not expect the clean localization of low energy degrees of freedom at the loci where $A(p)$ has small rank.

Suppose we try to proceed by ignoring this subtlety. Then we would have to conclude that the low energy theory localizes on the $R \leq N - 4$ locus which we denote by $Z_{[N]}$. This locus has dimension $N - 7$. The central charge for the degrees of freedom within $Z_{[N]}$ is bounded to be $\leq N - 7$, which falls short of the expected central charge $\widehat{c} = N - 4$ by at least three. This means that we need extra massless degrees of freedom from the modes transverse to $Z_{[N]}$. In other words, there is no separation of energy scales and the Born-Oppenheimer approximation is out of question.

To conclude, the Born-Oppenheimer approximation is not valid for the case of even N , and we do not yet know the nature of the $r \ll 0$ phase.

Acknowledgments

We would like to thank Victor Batyrev, Andrei Caldararu, Charles Doran, Nick Dorey, Ionut Ciocan-Fontanine, Sean Hartnoll, Bumsig Kim, Alexander Kuznetsov, David Morrison, Sunil Mukhi, Erich Poppitz, Cobi Sonnenschein, Duco van Straten and Edward Witten for several useful discussions and correspondences.

Both authors are grateful to the Research Institute for Mathematical Sciences and the Yukawa Institute for Theoretical Physics, Kyoto University, for their kind support during the visit in the summer of 2004 where a major part of this work was done, and to the Institute for Theoretical Physics in Amsterdam for the 2006 Summer Workshop which motivated us to write up a paper. K.H. also thanks the Banff International Research Station where he was introduced to Rødland’s work by Duco van Straten, and the Fields Institute and the Perimeter Institute for useful discussions with participants to the 2004-5 Thematic Program on the Geometry of String Theory.

K.H. is supported also by NSERC and Alfred P. Sloan Foundation. D.T. is supported by the Royal Society.

References

- [1] V.V. Batyrev, I. Ciocan-Fontanine, B. Kim and D. van Straten, *Conifold transitions and mirror symmetry for Calabi-Yau complete intersections in grassmannians*, *Nucl. Phys.* **B 514** (1998) 640.

- [2] E.A. Rødland, *The pfaffian Calabi-Yau, its mirror, and their link to the grassmannian $G(2,7)$* , *Composito Math.* **122** (2000) 135 [[math.AG/9801092](#)].
- [3] E. Witten, *Phases of $N = 2$ theories in two dimensions*, *Nucl. Phys.* **B 403** (1993) 159 [[hep-th/9301042](#)].
- [4] E. Witten, *Constraints on supersymmetry breaking*, *Nucl. Phys.* **B 202** (1982) 253.
- [5] A. Hanany and K. Hori, *Branes and $N = 2$ theories in two dimensions*, *Nucl. Phys.* **B 513** (1998) 119 [[hep-th/9707192](#)].
- [6] E. Witten, *Some comments on string dynamics*, [hep-th/9507121](#).
- [7] E. Witten, *On the conformal field theory of the Higgs branch*, *JHEP* **07** (1997) 003 [[hep-th/9707093](#)].
- [8] O. Aharony and M. Berkooz, *Ir dynamics of $D = 2$, $N = (4,4)$ gauge theories and DLCQ of 'little string theories'*, *JHEP* **10** (1999) 030 [[hep-th/9909101](#)].
- [9] E. Witten, *The Verlinde algebra and the cohomology of the grassmannian*, [hep-th/9312104](#) in *Geometry, topology and physics* **357**, *Conf. Proc. Lecture Notes Geom. Topology IV*, Internat. Press, Cambridge (1995).
- [10] P.S. Aspinwall, B.R. Greene and D.R. Morrison, *Calabi-Yau moduli space, mirror manifolds and spacetime topology change in string theory*, *Nucl. Phys.* **B 416** (1994) 414 [[hep-th/9309097](#)].
- [11] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil and E. Zaslow, *Mirror symmetry*, Clay Mathematics Monographs **1**, AMS, Cambridge (2003).
- [12] A. D'Adda, A.C. Davis, P. Di Vecchia and P. Salomonson, *An effective action for the supersymmetric CP^{N-1} model*, *Nucl. Phys.* **B 222** (1983) 45.
- [13] D.R. Morrison and M. Ronen Plesser, *Summing the instantons: quantum cohomology and mirror symmetry in toric varieties*, *Nucl. Phys.* **B 440** (1995) 279 [[hep-th/9412236](#)].
- [14] S.R. Coleman, *More about the massive Schwinger model*, *Ann. Phys. (NY)* **101** (1976) 239.
- [15] P. Candelas, X.C. De La Ossa, P.S. Green and L. Parkes, *A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory*, *Nucl. Phys.* **B 359** (1991) 21.
- [16] K. Hori and C. Vafa, *Mirror symmetry*, [hep-th/0002222](#).
- [17] K. Hori, A. Iqbal and C. Vafa, *D-branes and mirror symmetry*, [hep-th/0005247](#).
- [18] V.V. Batyrev and D. van Straten, *Generalized hypergeometric functions and rational curves on Calabi-Yau complete intersections in toric varieties*, *Commun. Math. Phys.* **168** (1995) 493 [[alg-geom/9307010](#)].
- [19] A. Ligober and J. Teitelbaum, *Lines on Calabi-Yau complete intersections, mirror symmetry, and Picard-Fuchs equations*, *Duke Math. J.* **1** (1993) 29.
- [20] I. Affleck, M. Dine and N. Seiberg, *Dynamical supersymmetry breaking in supersymmetric QCD*, *Nucl. Phys.* **B 241** (1984) 493.
- [21] J. Giedt, *Deconstruction and other approaches to supersymmetric lattice field theories*, *Int. J. Mod. Phys. A* **21** (2006) 3039 [[hep-lat/0602007](#)].

- [22] J. de Boer, K. Hori, Y. Oz and Z. Yin, *Branes and mirror symmetry in $N = 2$ supersymmetric gauge theories in three dimensions*, *Nucl. Phys. B* **502** (1997) 107 [[hep-th/9702154](#)].
- [23] J. de Boer, K. Hori and Y. Oz, *Dynamics of $N = 2$ supersymmetric gauge theories in three dimensions*, *Nucl. Phys. B* **500** (1997) 163 [[hep-th/9703100](#)].
- [24] O. Aharony, A. Hanany, K.A. Intriligator, N. Seiberg and M.J. Strassler, *Aspects of $N = 2$ supersymmetric gauge theories in three dimensions*, *Nucl. Phys. B* **499** (1997) 67 [[hep-th/9703110](#)].
- [25] I. Affleck, J.A. Harvey and E. Witten, *Instantons and (super)symmetry breaking in $(2 + 1)$ -dimensions*, *Nucl. Phys. B* **206** (1982) 413.
- [26] W. Lerche, C. Vafa and N.P. Warner, *Chiral rings in $N = 2$ superconformal theories*, *Nucl. Phys. B* **324** (1989) 427.
- [27] E. Witten, *On the Landau-Ginzburg description of $N = 2$ minimal models*, *Int. J. Mod. Phys. A* **9** (1994) 4783 [[hep-th/9304026](#)].
- [28] E. Silverstein and E. Witten, *Global $U(1)$ R -symmetry and conformal invariance of $(0, 2)$ models*, *Phys. Lett. B* **328** (1994) 307 [[hep-th/9403054](#)].
- [29] K. Hori and A. Kapustin, *Duality of the fermionic 2D black hole and $N = 2$ liouville theory as mirror symmetry*, *JHEP* **08** (2001) 045 [[hep-th/0104202](#)].
- [30] E.J. Martinec, *Algebraic geometry and effective lagrangians*, *Phys. Lett. B* **217** (1989) 431.
- [31] C. Vafa and N.P. Warner, *Catastrophes and the classification of conformal theories*, *Phys. Lett. B* **218** (1989) 51.
- [32] A.B. Zamolodchikov and V.A. Fateev, *Disorder fields in two-dimensional conformal quantum field theory and $N = 2$ extended supersymmetry*, *Sov. Phys. JETP* **63** (1986) 913.
- [33] K. Hori and D. Tong, *Summing the non-abelian instantons* (tentative title), to appear.
- [34] N. Seiberg, *Exact results on the space of vacua of four-dimensional SUSY gauge theories*, *Phys. Rev. D* **49** (1994) 6857 [[hep-th/9402044](#)].
- [35] C. Vafa, *Modular invariance and discrete torsion on orbifolds*, *Nucl. Phys. B* **273** (1986) 592.
- [36] E. Witten (notes by J. Morgan and D. Morrison), lecture 15 in: *Quantum fields and strings: a course for mathematicians*, **2**, P. Deligne et al eds., AMS, Princeton (1999).
- [37] N. Seiberg, *Electric-magnetic duality in supersymmetric nonabelian gauge theories*, *Nucl. Phys. B* **435** (1995) 129 [[hep-th/9411149](#)].
- [38] P.H. Ginsparg, *Applied conformal field theory in Les Houches Summer School 1988*, [hep-th/9108028](#).
- [39] Y. Kazama and H. Suzuki, *New $N = 2$ superconformal field theories and superstring compactification*, *Nucl. Phys. B* **321** (1989) 232.
- [40] A. Kuznetsov, *Homological projective duality for grassmannians of lines*, [math.AG/0610957](#).
- [41] A. Kuznetsov, *Homological projective duality*, [math.AG/0507292](#).
- [42] L. Borisov and A. Caldararu, *The pfaffian-grassmannian derived equivalence*, [math.AG/0608404](#).
- [43] M. Herbst, K. Hori and D. Page, *Phases of $\mathcal{N} = 2$ theories in $1 + 1$ dimensions with boundary* (tentative title), to appear.